Set Partition Modulation

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***Abstract*—In this paper, we propose a novel modulation scheme called set partition modulation (SPM). In this scheme, set partitioning and ordered subsets in the set partitions are used to form codewords. We define different SPM variants and depict a practical model for using SPM with orthogonal frequency division multiplexing (OFDM). For the OFDM-SPM schemes, different constellations are used to distinguish between different subsets in a set partition. It is shown that the proposed SPM variants are general schemes, which encompass multi-mode OFDM with index modulation (MM-OFDM-IM) and dual-mode OFDM with index modulation (DM-OFDM-IM) as special cases. It is also shown that OFDM-SPM schemes are capable of exhibiting better error performance than conventional OFDM, OFDM-IM, DM-OFDM-IM, and MM-OFDM-IM.**

***Index Terms*—Orthogonal frequency division multiplexing (OFDM), index modulation, set partitions.**

I. Introduction

The idea of encoding information in the order of the elements in a codeword was first considered by Slepian in [1]. Slepian's idea, which he called *permutation modulation* (PM), was based on forming a codebook by permuting elements of a codeword. As a subclass of PM, *index modulation* (IM) techniques have attracted exceptional interest due to their promising advantages including better error performance and improved energy/spectral efficiency [2]. The basic idea of IM lies in the encoding of information the indices of the active sources. For example, in multi-antenna communication techniques such as *spatial modulation* (SM) [3], IM codewords are realized by combinations of the (de)activated transmit antenna indices. In contrast, IM applied to orthogonal frequency-division multiplexing (OFDM) transmissions relies on the activation patterns of the *subcarriers* to encode information [4]–[6]. Other PM/IM-based applications exist with manifestations in space, time and frequency (see, e.g., [7]).

With OFDM-IM, in order to embed information in combinations of the subcarriers, a certain amount of subcarriers are deactivated. Although carrying information on the combinations of activated/deactivated subcarriers results in better error performance and improved spectral/energy efficiency for small modulation orders, it is difficult to obtain spectral efficiencies comparable to conventional OFDM for high modulation orders. In order to overcome this problem, researchers have devised the idea of using distinguishable constellations on subcarriers rather than deactivating them [8]–[12]. In [8], two distinguishable constellations are employed in an OFDM scheme called *dual mode OFDM with IM* (DM-OFDM-IM) to utilize the combinations of distinguishable constellation symbols on subcarriers rather than turning subcarriers off. *Generalized DM-OFDM-IM*, which alters the number of subcarriers modulated by the same constellation, is proposed in [9]. In [10], in addition to two distinguishable constellations, some subcarriers are allowed to remain unused, which yields a third mode that, as it turns out, enhances the bit-error rate (BER) performance of DM-OFDM-IM and OFDM-IM. In [11], Wen et al. propose a *multi-mode OFDM-IM* (MM-OFDM-IM) scheme, which uses distinguishable constellations on each subcarrier of OFDM sub-blocks to increase the spectral efficiency as well as to improve BER performance. In [12] and [13], the MM-OFDM-IM scheme is further generalized.

Against this background, we develop a new codebook design method, which we call *set partition modulation* (SPM). The proposed method uses partitions of codeword elements rather than permutations or combinations of such elements to encode information. We define different variants of SPM and give a practical model for applying SPM with OFDM transmissions. The proposed OFDM-SPM schemes employ distinguishable constellations, similar to MM-OFDM-IM; however, in OFDM-SPM, the different constellations are used to distinguish between subsets in a set partition, unlike MM-OFDM-IM where distinguishable constellations are used to construct permutations. The achievable data rate and BER of OFDM-SPM variants are investigated in this paper, and an upper bound on the BER is obtained. Our analytical findings show that OFDM-SPM variants are capable of outperforming conventional OFDM, OFDM-IM, DM-OFDM-IM and MM-OFDM-IM in terms of data rate and BER.

The rest of the paper is organized as follows. In Section II, we define SPM and its variants. OFDM-SPM is described in Section III. Performance analysis is undertaken in Section IV. We present and compare analytical and numerical results in Section V. Finally, we conclude the paper in Section VI.

II. Set Partition Modulation

In this section, we describe the basic idea of SPM. We begin with some useful definitions and relations.

Definition 1. Set Partition: A set partition is the grouping the elements of a set in a way that the groups are disjoint and the union of the groups gives the set.

Definition 2. Stirling Number of the Second Kind: The Stirling number of the second kind, denoted by $\left\{ {N \atop K} \right\}$, can be defined as the number of ways to partition an N-element set into K non-empty subsets.

Definition 3. Bell Number: The Bell number B_N enumerates the total number of partitions of a set of N elements. The Bell number is related to Stirling numbers of the second kind through the equation $B_N = \sum_{K=1}^{N} \left\{ {N \atop K} \right\}$.

Definition 4. Ordered Bell Number: The ordered Bell number $\breve{B}_N$ enumerates the total number of partitions of an N-element set considering all permutations of subsets for each partition. The ordered Bell number satisfies the equation $\breve{B}_N = \sum_{K=1}^{N} K! \left\{ {N \atop K} \right\}$.

A. SPM

In an SPM system, a codebook of L codewords $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_L$ is constructed such that each codeword is a sequence of N elements, which are drawn from a constellation diagram in the complex plane, i.e., $\mathbf{x}_l = \{x_{l1}, x_{l2}, \ldots, x_{lN}\}$, $l = 1, \ldots, L$, where $x_{ln} \in \mathbb{C}$, $n \in \{1, \ldots, N\}$, is an M-ary symbol. Each codeword is mapped to a partition of an N-element set $\mathcal{X}$ into K subsets. Since the number of ways one can form the partition is $\left\{ {N \atop K} \right\}$, the SPM codebook size is given by $L_{SPM} = \left\{ {N \atop K} \right\}$. We call $\mathcal{X}$ the generator set of an SPM codeword. To produce unique codewords for SPM, an N-element codeword should have at least K distinguishable elements, and each distinguishable element, $\mu_k, k = 1, \ldots, K$, is used to specify which element in the codewords belongs to which subset. Therefore, distinguishable elements differentiate each subset from the other(s).

Example 1. As an example, consider the ways to partition a four-element set $\mathcal{X} \coloneqq \{x_1, x_2, x_3, x_4\}$ into two-element subsets. This is shown along with the corresponding SPM codewords in Table I. As seen from the table, to obtain the codewords in an SPM codebook, we first partition the elements of the generator set $\mathcal{X}$ into two-element subsets $\mathcal{S}_i$, $i = 1, \ldots, L$ where $L = 7$[1]. Then, we use the subset identifiers μ_1 and μ_2 to represent the elements that belong to different subsets. For example, for the first codeword in Table I, we have the partition $\mathcal{S}_1 \coloneqq \{\{x_1\}, \{x_2, x_3, x_4\}\}$. Since the element x_1 and the elements x_2, x_3, and x_4 are in the first and second subsets, respectively, we assign μ_1 to first element and μ_2 to remaining elements.

[1]Note that $\left\{ {4 \atop 2} \right\} = 7$.

Table I
SET PARTITIONING AND SPM CODEWORD GENERATION EXAMPLE FOR $N = 4$, $K = 2$.

Set Partitions of $\mathcal{X}$	SPM Codeword
$\mathcal{S}_1 = \{\{x_1\}, \{x_2, x_3, x_4\}\}$	$\mathbf{x}_1 = \{\mu_1, \mu_2, \mu_2, \mu_2\}$
$\mathcal{S}_2 = \{\{x_2\}, \{x_1, x_3, x_4\}\}$	$\mathbf{x}_2 = \{\mu_2, \mu_1, \mu_2, \mu_2\}$
$\mathcal{S}_3 = \{\{x_3\}, \{x_1, x_2, x_4\}\}$	$\mathbf{x}_3 = \{\mu_2, \mu_2, \mu_1, \mu_2\}$
$\mathcal{S}_4 = \{\{x_4\}, \{x_1, x_2, x_3\}\}$	$\mathbf{x}_4 = \{\mu_2, \mu_2, \mu_2, \mu_1\}$
$\mathcal{S}_5 = \{\{x_1, x_2\}, \{x_3, x_4\}\}$	$\mathbf{x}_5 = \{\mu_1, \mu_1, \mu_2, \mu_2\}$
$\mathcal{S}_6 = \{\{x_1, x_3\}, \{x_2, x_4\}\}$	$\mathbf{x}_6 = \{\mu_1, \mu_2, \mu_1, \mu_2\}$
$\mathcal{S}_7 = \{\{x_1, x_4\}, \{x_2, x_3\}\}$	$\mathbf{x}_7 = \{\mu_1, \mu_2, \mu_2, \mu_1\}$

Table II
SET PARTITIONING AND OSPM CODEWORD GENERATION EXAMPLE FOR $N = 3$, $K = 2$.

Set Partitions of $\mathcal{X}$	SPM Codeword
$\mathcal{S}_1 = \{\{x_1\}, \{x_2, x_3\}\}$	$\mathbf{x}_1 = \{\mu_1, \mu_2, \mu_2\}$
$\mathcal{S}_2 = \{\{x_2\}, \{x_1, x_3\}\}$	$\mathbf{x}_2 = \{\mu_2, \mu_1, \mu_2\}$
$\mathcal{S}_3 = \{\{x_3\}, \{x_1, x_2\}\}$	$\mathbf{x}_3 = \{\mu_2, \mu_2, \mu_1\}$
$\mathcal{S}_4 = \{\{x_2, x_3\}, \{x_1\}\}$	$\mathbf{x}_4 = \{\mu_2, \mu_1, \mu_1\}$
$\mathcal{S}_5 = \{\{x_1, x_3\}, \{x_2\}\}$	$\mathbf{x}_5 = \{\mu_1, \mu_2, \mu_1\}$
$\mathcal{S}_6 = \{\{x_1, x_2\}, \{x_3\}\}$	$\mathbf{x}_6 = \{\mu_1, \mu_1, \mu_2\}$

B. Ordered SPM

Ordered SPM (OSPM) is an extended version of SPM in which the codebook size is increased by considering permutations of subsets in a partition. Hence, the codebook size of OSPM is given by $L_{OSPM} = K! \left\{ {N \atop K} \right\}$. This simple extension is best illustrated with an example.

Example 2. In Table II, we give an example of a mapping between set partitions and OSPM codewords for $N = 3$ and $K = 2$. As seen from the table, the first three codewords are SPM codewords and we further obtain an extended codebook by taking the permutations of the subsets in a set partition into account. For this example, by exhibiting additional codewords obtained from the order of the subsets, we end up with $2! \left\{ {3 \atop 2} \right\} = 6$ codewords.

C. Full SPM

In Full SPM (FSPM), the codewords $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{L_{FSPM}}$ are generated by partitioning an N-element set $\mathcal{X}$ into non-empty disjoint subsets in such a way that the number of these subsets takes any possible value, $K \in \{1, \ldots, N\}$. In other words, all partitions of an N-element set $\mathcal{X}$ into non-empty disjoint subsets are used to form the FSPM codebook. For FSPM, the codebook size is equal to the Bell number B_N, i.e., $L_{FSPM} = B_N = \sum_{K=1}^{N} \left\{ {N \atop K} \right\}$.

Example 3. Let us consider an example of how we define the codewords in an FSPM codebook when $N = 3$. Partitions of a three-element set $\mathcal{X} \coloneqq \{x_1, x_2, x_3\}$ is given in Table II along with the corresponding FSPM codewords. As seen from the table, to obtain the codewords in an FSPM codebook, we first partition

Table III
SET PARTITIONING AND FSPM CODEWORD GENERATION EXAMPLE FOR $N = 3$.

Set Partitions of $\mathcal{X}$	SPM Codeword
$\mathcal{S}_1 = \{\{x_1, x_2, x_3\}\}$	$\mathbf{x}_1 = \{\mu_1, \mu_1, \mu_1\}$
$\mathcal{S}_2 = \{\{x_1\}, \{x_2, x_3\}\}$	$\mathbf{x}_2 = \{\mu_1, \mu_2, \mu_2\}$
$\mathcal{S}_3 = \{\{x_2\}, \{x_1, x_3\}\}$	$\mathbf{x}_3 = \{\mu_2, \mu_1, \mu_2\}$
$\mathcal{S}_4 = \{\{x_3\}, \{x_1, x_2\}\}$	$\mathbf{x}_4 = \{\mu_2, \mu_2, \mu_1\}$
$\mathcal{S}_5 = \{\{x_1\}, \{x_2\}, \{x_3\}\}$	$\mathbf{x}_5 = \{\mu_1, \mu_2, \mu_3\}$

the elements of the generator set $\mathcal{X}$ into subsets $\mathcal{S}_i$, $i = 1, \ldots, L_{FSPM}$ where $L_{FSPM} = 5$. Note that Bell number for $N = 3$ is $B_3 = 5$. Then, we use the subset identifier μ_k, $k \in \{1, \ldots, K\}$, to represent the elements that belong to the kth subset.

D. Ordered Full SPM

We can further increase the number of codewords in an FSPM codebook by considering the permutations of the subsets in a partition. In this regard, we define ordered full SPM (OFSPM) as a modulation scheme that forms its codebook by using all partitions of an N-element set $\mathcal{X}$ along with all permutations of the subsets in each partition. Hence, the OFSPM codebook size is given by the ordered Bell number $\breve{B}_N$, i.e., $L_{OFSPM} = \breve{B}_N = \sum_{k=1}^{N} k!\left\{{N \atop k}\right\} = \breve{B}_N$.

III. PRACTICAL MODEL FOR OFDM

We present a practical system model in which we apply SPM schemes to OFDM transmissions. The transmitter structure of the OFDM-SPM scheme is shown in Fig. 1. In this scheme, m input bits enter the SPM transmitter, and these bits are divided into $B = m/f$ blocks, each having f input bits. Similarly, the total number of subcarriers N_T is also divided into $B = N_T/N$ blocks, each having N subcarriers. For each block of input bits, f information bits are modulated by an SPM encoder and the resulting modulated symbols are carried by N subcarriers.

Since each bit and each subcarrier block have the same mapping operation, we focus on a single block, the bth block (where $b \in \{1, 2, \ldots, B\}$), in what follows. In the bth block, the SPM encoder further divides f information bits into two parts, one of them having f_1 bits and the other one having f_2 bits with $f_1 + f_2 = f$. The first f_1 bits are used to determine the specific set partition $\mathcal{S}_i^b$, $i = 1, \ldots, L$ ($L \in \{L_{SPM}, L_{OSPM}, L_{FSPM}, L_{OFSPM}\}$), of the N-element generator set $\mathcal{X} := \{x_1, x_2, \ldots, x_N\}$ belonging to the one of the variants of SPM defined above. The chosen partition is mapped to the corresponding SPM codeword $\mathbf{x}_i^b$ where the superscript b stands for the bth block. Here, each element in the SPM codeword corresponds to a subcarrier in the bth block. The remaining f_2 bits are used to modulate symbols on the N subcarriers, considering the corresponding mapping of the set partition determined by the first f_1 bits. As discussed in the previous subsection, we use different subset identifiers in order to produce unique SPM codewords. To preserve the uniqueness property of SPM codewords and modulate the symbols on each subcarrier, we further assume that each subset identifier μ_k is an element of a disjoint M-ary signal constellation $\mathcal{M}_k$, i.e., $\mu_k \in \mathcal{M}_k$ and $\mathcal{M}_k \cap \mathcal{M}_{\hat{k}} = \emptyset$, where $k, \hat{k} \in \{1, 2, \ldots, N\}$ and $k \neq \hat{k}$[2]. For convenience, we choose the size of each constellation as M and, therefore, $f_2 = N \log_2 M$. By following the useful design guidelines in [11], we obtain the distinguishable constellations $\mathcal{M}_k$ by rotating each constellation with the angle of $(k-1)\pi/N$, $k = 1, \ldots, N$, to maximize the distance between constellation points.

Table IV
A LOOK-UP TABLE EXAMPLE CORRESPONDING TO A BIT-TO-PARTITION MAPPING FOR SPM ($N = 4$ AND $K = 2$).

f_1 bits	Set Partitions of $\mathcal{X}$
[0 0]	$\mathcal{S}_1^b = \{\{x_1\}, \{x_2, x_3, x_4\}\}$
[0 1]	$\mathcal{S}_2^b = \{\{x_2\}, \{x_1, x_3, x_4\}\}$
[1 0]	$\mathcal{S}_3^b = \{\{x_3\}, \{x_1, x_2, x_4\}\}$
[1 1]	$\mathcal{S}_4^b = \{\{x_4\}, \{x_1, x_2, x_3\}\}$

The mapping of f_1 bits to the set partitions can be performed by using a look-up table in a similar manner as was proposed to map information bits to subcarrier activation patterns in [4], or to permutation indices as detailed in [11]. A look-up table example illustrating the mapping of f_1 bits to the set partitions is given in Table IV for $N = 4$ and $K = 2$. Note that we are only able to use 2^{f_1} set partitions.[3] As seen from the table, f_1 bits are used to determine the specific set partition at first. The chosen set partition is then used to determine the SPM codeword as discussed earlier.

Once the mapping between f_1 bits and set partitions has been completed, $f_2 = N \log_2 M$ bits are used to determine the modulated symbols, or in other words subset identifiers, on each subcarrier. Hence, one of the SPM codewords $\mathbf{x}^b \in \{\mathbf{x}_1^b, \ldots, \mathbf{x}_L^b\}$ along with the corresponding modulation symbols $\{\mu_k\}$ constitutes the symbol vector of the bth block. After obtaining symbol vectors for all blocks, an OFDM block creator forms the overall symbol vector $\mathbf{x} := [x(1), x(2), \ldots, x(N_T)] = [\mathbf{x}^1, \ldots, \mathbf{x}^b, \ldots, \mathbf{x}^B]^T \in \mathcal{C}^{N_T \times 1}$. Here, we assume that each element of $\mathbf{x}$ is distributed among equally spaced

[2]Note that the subset identifiers are not necessarily elements of disjoint M-ary signal constellations and each of them may be chosen as a single constellation point in the same M-ary constellation. Hence, assigning an unique constellation point to each subset identifier would be enough to constitute an SPM scheme. However, in this special case, the number of information bits transmitted by an OFDM block is decreased by f_2 bits since f_2 bits are not used to modulate the subset identifiers.

[3]It is possible to utilize all set partitions by employing binary coding algorithms [14], [15].

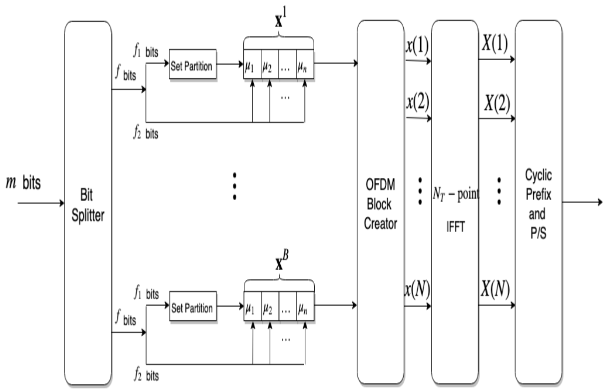

Figure 1. Transmitter structure of OFDM-SPM scheme.

subcarriers to ensure orthogonality in frequency, and each modulated symbol carried by a subcarrier has unit energy, i.e., $\mathrm{E}[|x(t)|^2] = 1$, $t = 1, \ldots, N_T$. After this point, exactly the same operations as conventional OFDM are applied. The symbol vector is processed with an N_T-point IFFT, and a cyclic prefix of sufficient length, which is not lower than the memory of the discrete channel impulse response, is attached to the beginning of each time-domain symbol vector. After parallel-to-serial and up-conversion, transmission is operated over a frequency-selective Rayleigh fading channel.

At the receiver, the received signal is down converted and the cyclic prefix is then removed from each received baseband symbol vector before processing with an FFT. After employing an N_T-point FFT operation, the frequency domain received signal vector can be written as

$$\mathbf{y} \coloneqq [y(1), y(2), \ldots, y(N_T)]^T = \sqrt{E_S}\mathbf{X}\mathbf{h} + \mathbf{n} \tag{1}$$

where E_S is the energy of the transmitted symbol vector and $\mathbf{X} = \mathrm{diag}(\mathbf{x})$. Moreover, $\mathbf{h}$ and $\mathbf{n}$ are $N_T \times 1$ channel and noise vectors, respectively. Elements of these vectors follow the complex-valued Gaussian distributions $\mathcal{CN}(0, 1)$ and $\mathcal{CN}(0, N_0)$, respectively, where N_0 is the noise variance.

Since the encoding procedure for each block is independent of others, decoding can be performed independently at receiver. Hence, using maximum likelihood (ML) detection, the detected symbol vector for the bth block can be written as

$$\hat{\mathbf{x}}^b = \arg\min_{\mathcal{S}_i, \mu_k} ||\mathbf{y}^b - \sqrt{E_S}\mathbf{X}^b\mathbf{h}^b||^2 \tag{2}$$

where $\mathbf{y}^b = [y((b-1)N+1), \ldots, y(bN)]^T$, $\mathbf{X}^b = \mathrm{diag}(\mathbf{x}^b)$ and $\mathbf{h}^b = [h((b-1)N+1), \ldots, h(bN)]^T$.

IV. Performance Analysis

In this section, we analyze the data rate and bit-error rate (BER) of the proposed SPM schemes.

A. Data Rate

We analyze the data rate of the proposed OFDM-SPM schemes in terms of number of bits corresponding to the OFDM-SPM codewords transmitted per subcarrier. Here, we do not take cyclic prefix length into account for convenience. We also provide some useful expressions for the number of partitions obtained by SPM schemes along with comparisons regarding OFDM benchmarks. We assume $f_2 = N \log_2 M$ for each SPM variant.

1) OFDM-SPM: As discussed in Section II, the number of set partitions produced by SPM is given by the Stirling number of the second kind $\left\{ {N \atop K} \right\}$. Considering the fact that each symbol on each subcarrier has the modulation order M, the achievable data rate per subcarrier for an OFDM-SPM scheme with N subcarriers and K subset identifiers in each sub-block is

$$R_{SPM} = \frac{f_1 + f_2}{N} = \frac{\lfloor \log_2 \left\{ {N \atop K} \right\} \rfloor + N \log_2 M}{N} \tag{3}$$

where $\lfloor . \rfloor$ is the floor operation.

The Stirling number of the second kind, $\left\{ {N \atop K} \right\}$, can be written as [16]

$$\left\{ {N \atop K} \right\} = \frac{1}{K!} \sum_{j=0}^{K} (-1)^j \binom{K}{j} (K-j)^N. \tag{4}$$

It is also straightforward to show that the following recurrence holds:

$$\left\{ {N \atop K} \right\} = K \left\{ {N-1 \atop K} \right\} + \left\{ {N-1 \atop K-1} \right\}. \tag{5}$$

Remark. Consider an OFDM-SPM block having two distinguishable constellations, i.e., $K = 2$, on N subcarriers and a DM-OFDM-IM block having two distinguishable constellations on N subcarriers in which $(N-d)$ of the subcarriers is modulated by one of two different constellations and the remaining d of them is modulated by the other constellation.[4] Except for the case where $N = 2$ and $d = 1$, the number of set partitions obtained by the OFDM-SPM encoder is equal to or greater than the number of subcarrier combinations obtained by the DM-OFDM-IM encoder, i.e., $\left\{ {N \atop 2} \right\} \geq \binom{N}{d}$ for $N \geq 2$. The equality holds for $N = 2$ and $d = 2$. This means that the achievable data rate for OFDM-SPM is equal to or greater than that of DM-OFDM-IM when the subcarriers of both schemes carry symbols that have the same modulation order. One may check that $\left\{ {N \atop 2} \right\} = 2^{N-1} - 1$. Moreover, from the recurrence relation of the binomial coefficient, we have $\binom{N}{d} = \binom{N-1}{d} + \binom{N-1}{d-1}$. If we compare $\binom{N}{d}$ with $\left\{ {N \atop 2} \right\}$, we see that $\left\{ {N \atop 2} \right\} \geq \binom{N}{d}$ for $N \geq 2$ except for the case where $N = 2$ and $d = 1$.

[4]It is fair to compare these two schemes since both of the schemes has two distinguishable constellations.

2) OFDM-OSPM: The achievable data rate per subcarrier for an OFDM-OSPM scheme having N subcarriers and K subset identifiers in each sub-block can be written as

$$R_{OSPM} = \frac{f_1 + f_2}{N} = \frac{\lfloor \log_2 K! \left\{ {N \atop K} \right\} \rfloor + N \log_2 M}{N} \tag{6}$$

Remark. It is straightforward to show that $2! \left\{ {N \atop 2} \right\} \geq \binom{N}{d}$. However, it is important to note that an OFDM-OSPM codebook, which incorporates $K = 2$ element partitions and their ordered counterparts, subsumes a DM-OFDM-IM codebook. This can easily be proved by considering set partitions along with permutations when $K = 2$. Hence, it can be concluded that OFDM-OSPM is a more general scheme, which encompasses the subcarrier combinations generated by a DM-OFDM-IM encoder. It is also important to note that an OFDM-OSPM encoder produces the same index patterns as a MM-OFDM-IM encoder for $K = N$. Hence, MM-OFDM-IM is a special case of OFDM-OSPM when $K = N$. Moreover, the partitions of an N-element set into two subsets would result in the same index symbols as GDM-OFDM-IM when the set, $\mathcal{K}$, containing possible numbers of subcarriers having one of a number of distinguishable constellations in each OFDM sub-block is given by $\mathcal{K} = \{1, 2, \ldots, N-1\}$ for GDM-OFDM-IM. *Despite these similarities to IM schemes, it is important to recognize that OSPM is inherently different due to the use of set partitions to encode information instead of index patterns or permutations.*

3) OFDM-FSPM: The achievable data rate per subcarrier for an OFDM-FSPM scheme having N subcarriers can be written as

$$R_{FSPM} = \frac{f_1 + f_2}{N} = \frac{\lfloor \log_2 B_N \rfloor + N \log_2 M}{N}. \tag{7}$$

4) OFDM-OFSPM: The achievable data rate per subcarrier for an OFDM-OFSPM scheme having N subcarriers can be written as

$$R_{OFSPM} = \frac{f_1 + f_2}{N} = \frac{\lfloor \log_2 \breve{B}_N \rfloor + N \log_2 M}{N}. \tag{8}$$

Remark. It is also straightforward to show that $\breve{B}_N > N!$ for $N \geq 2$, since the ordered set partitions include all permutations of N-element set partitions and the number of the permutations of N-element partitions is equal to $N!$. In other words, for an N-element set $\mathcal{X}$ with $K = N$, $\mathcal{S} = \{\{x_1\}, \{x_2\}, \ldots, \{x_N\}\}$ is a valid set partition, and ordering the subsets of this set would result in $N!$ different partitions. Hence, the resulting OFDM-OFSPM codebook contains the codeword $\mathbf{x}^b = \{\mu_1, \mu_2, \ldots, \mu_N\}$ along with the codewords representing all permutations of the elements of $\mathbf{x}^b$. Note that these codewords form the MM-OFDM-IM codebook, and OFDM-OFSPM is a more general scheme compared to MM-OFDM-IM since it subsumes the all codewords formed by an MM-OFDM-IM encoder.

B. Bit-Error Rate

$P(\mathbf{X}^i \to \mathbf{X}^j)$ is the pairwise error probability (PEP) associated with the erroneous detection of $\mathbf{X}^i$ as $\mathbf{X}^j$ where $\mathbf{X}^i = \mathrm{diag}(\mathbf{x}^i)$ and $\mathbf{X}^j = \mathrm{diag}(\mathbf{x}^j)$. From (2), the PEP conditioned on the channel coefficients is given by

$$P(\mathbf{X}^i \to \mathbf{X}^j | \mathbf{h}) = Q\left(\sqrt{\frac{E_S ||(\mathbf{X}^i - \mathbf{X}^j)\mathbf{h}||^2}{2N_0}} \right). \tag{9}$$

By using the identity $Q(x) \approx \frac{1}{12} e^{-x^2/2} + \frac{1}{4} e^{-2x^2/3}$ and averaging over $\mathbf{h}$, an approximate unconditional PEP expression can be obtained [4]:

$$\begin{aligned} P(\mathbf{X}^i \to \mathbf{X}^j) &= \mathrm{E}_{\mathbf{h}} \left[P(\mathbf{X}^i \to \mathbf{X}^j | \mathbf{h}) \right] \\ &\approx \frac{1/12}{\det(\mathbf{I}_N + \frac{E_S}{4N_0} \mathbf{C} \mathbf{Z}_{ij})} \\ &\quad + \frac{1/4}{\det(\mathbf{I}_N + \frac{E_S}{3N_0} \mathbf{C} \mathbf{Z}_{ij})} \end{aligned} \tag{10}$$

where $\mathbf{I}_N$ denotes the $N \times N$ identity matrix, $\mathbf{C} = \mathrm{E}_{\mathbf{h}}[\mathbf{h}\mathbf{h}^H]$ and $\mathbf{Z}_{ij} = (\mathbf{X}^i - \mathbf{X}^j)^H (\mathbf{X}^i - \mathbf{X}^j)$.

An upper bound on the average BER is given by the well-known union bound

$$P_b \leq \frac{1}{f 2^f} \sum_{i=1}^{2^f} \sum_{j=1}^{2^f} P(\mathbf{X}^i \to \mathbf{X}^j) D(\mathbf{X}^i \to \mathbf{X}^j) \tag{11}$$

where $D(\mathbf{X}^i \to \mathbf{X}^j)$ is the number of bits in error for the corresponding pairwise error event. Note that the upper bound expression given in (11) is valid for all OFDM-SPM schemes.

V. Numerical Results

A. Data Rate

In this subsection, we compare OFDM-SPM schemes with DM-OFDM-IM and MM-OFDM-IM in terms of the number of index bits transmitted per subcarrier. Since all the schemes considered in this section activate all subcarriers and the modulation order of the carried symbols on each subcarrier can be chosen to be the same, we ignore the modulation bits transmitted per subcarrier. Also, we do not restrict the codebook sizes to a power of two, since all the codewords in a codebook can be utilized by a binary coding technique such as Huffman coding regardless of the number of codewords [14], [15].

In Fig. 2, we compare the data rates of the proposed OFDM-SPM schemes with DM-OFDM-IM and MM-OFDM-IM. The data rate results in terms of the number of index bits per subcarrier are given as a function of N. In order to provide a fair comparison with DM-OFDM-IM, we allocate two distinguishable constellations to OFDM-SPM and OFDM-OSPM. Also, to reach the maximum number of index bits, DM-OFDM-IM modulates

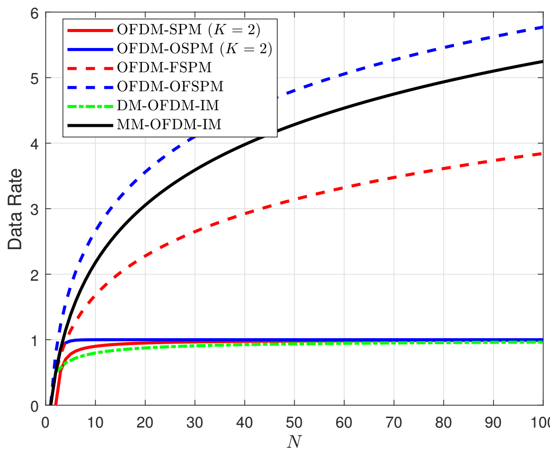

Figure 2. Data rate comparison of OFDM-SPM variants with DM-OFDM-IM and MM-OFDM-IM.

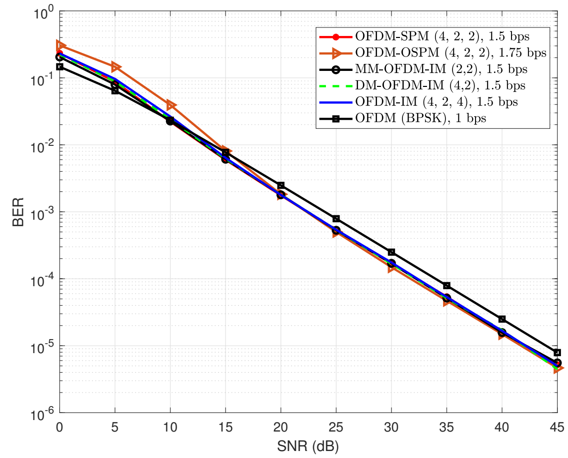

Figure 3. BER comparison of OFDM-SPM $(4, 2, 2)$ and OFDM-OSPM $(4, 2, 2)$ with MM-OFDM-IM $(2, 2)$, DM-OFDM-IM $(4, 2)$, OFDM-IM $(4, 2, 4)$ and OFDM (BPSK).

the half of the subcarriers by one of the distinguishable constellations and the other half by the other constellation. The data rate results verify the remarks in the previous section and indicate that OFDM-SPM and OFDM-OSPM outperform DM-OFDM-IM for all values of N except for $N = 2$. On the other hand, although MM-OFDM-IM considerably outperforms OFDM-FSPM, it is outperformed by OFDM-OFSPM for all values of N.

B. Bit Error Rate

In this section, we compare the proposed OFDM-SPM schemes with conventional OFDM and IM benchmarks in terms of BER performance. In Figs. 3 and 4, OFDM-SPM (N, K, M) and OFDM-OSPM (N, K, M) stand for OFDM-SPM schemes having N subcarriers and K distinguishable constellations in each OFDM sub-block along with M-PSK symbols on each subcarrier, whereas OFDM-FSPM (N, M) and OFDM-OFSPM (N, M) stand for variants of OFDM-SPM employing all set partitions and having N subcarriers with M-PSK symbols in each OFDM sub-block. Moreover, OFDM-IM (N, K_a, M) stands for the conventional OFDM-IM scheme having K_a activated subcarriers out of N in each sub-block and employing M-PSK modulation on the activated subcarriers. Finally, DM-OFDM-IM (N, M) signifies a dual-mode scheme having N subcarriers along with two distinguishable M-PSK constellations, and MM-OFDM-IM (N, M) represents a multi-mode scheme having N subcarriers along with N distinguishable M-PSK constellations in each sub-block.

In Fig. 3, we compare the BER performance of the proposed OFDM-SPM $(4, 2, 2)$ and OFDM-OSPM $(4, 2, 2)$ with conventional OFDM (BPSK), OFDM-IM $(4, 2, 4)$, DM-OFDM-IM $(4, 2)$ and MM-OFDM-IM $(2, 2)$ schemes. Except for OFDM-OSPM $(4, 2, 2)$ and conventional OFDM (BPSK), all schemes exhibit the same spectral efficiency, which is 1.5 bits per subcarrier (bps). Spectral efficiencies for OFDM-OSPM $(4, 2, 2)$ and conventional OFDM (BPSK) are 1.75 bps and 1 bps respectively. Here, OFDM-SPM $(4, 2, 2)$ and OFDM-OSPM $(4, 2, 2)$ encoders produce $\left\{ 4 \atop 2 \right\} = 7$ and $2!\left\{ 4 \atop 2 \right\} = 14$ codewords, and then $2^{\log_2 \lfloor 7 \rfloor} = 4$ and $2^{\log_2 \lfloor 14 \rfloor} = 8$ set partitions are selected by exhaustively searching among possible set partition codeword subsets according to the well known rank and determinant criteria [17]. As seen from the figure, OFDM-SPM $(4, 2, 2)$ exhibits almost the same BER performance as OFDM-IM $(4, 2, 4)$, DM-OFDM-IM $(4, 2)$, and MM-OFDM-IM $(2, 2)$, and it outperforms conventional OFDM (BPSK) at medium-to-high SNR. More importantly, although OFDM-OSPM $(4, 2, 2)$ has higher spectral efficiency than other schemes, it provides marginally better BER performance compared to other schemes at high SNR.

In Fig. 4, the proposed OFDM-FSPM $(4, 2)$ and OFDM-OFSPM $(4, 2)$ schemes are compared with conventional OFDM (QPSK), OFDM-IM $(4, 3, 4)$ and MM-OFDM-IM $(4, 2)$. Except for OFDM-FSPM $(4, 2)$ and OFDM-OFSPM $(4, 2)$, all schemes have the same spectral efficiency of 2 bps. Spectral efficiencies for OFDM-FSPM $(4, 2)$ and OFDM-OFSPM $(4, 2)$ are 1.75 bps and 2.25 bps respectively. Here, OFDM-FSPM $(4, 2)$ and OFDM-OFSPM $(4, 2)$ encoders produce $B_N = 15$ and $\breve{B}_N = 75$ codewords, respectively. Eight and 32 codewords are then selected for for OFDM-FSPM $(4, 2)$ and OFDM-OFSPM $(4, 2)$, respectively, in a way that the rank of the difference matrices for the codeword pairs in the selected codebook is at least two [11]. We also provide results pertaining to the theoretical upper bound for the OFDM-SPM schemes. As observed from the figure, upper-bound curves are consistent with computer simulations, especially at high SNR. Although OFDM-

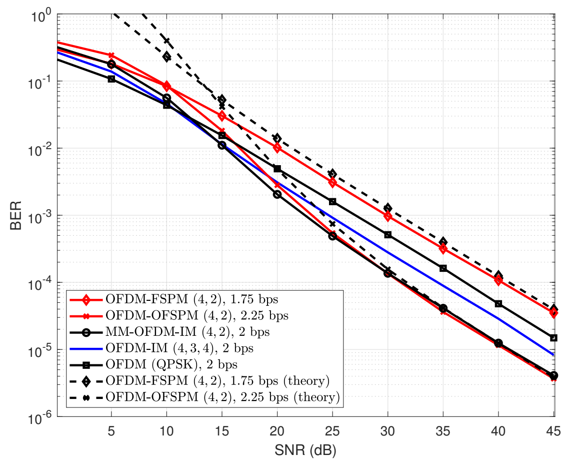

Figure 4. BER comparison of OFDM-FSPM $(4, 2)$ and OFDM-OFSPM $(4, 2)$ with MM-OFDM-IM $(4, 2)$, OFDM-IM $(4, 3, 4)$ and OFDM (QPSK).

FSPM $(4, 2)$ cannot provide a BER advantage relative to OFDM-IM $(4, 3, 4)$ and MM-OFDM-IM $(4, 2)$, OFDM-OFSPM $(4, 2)$ exhibits superior BER performance against all benchmarks at high SNR along with enhanced spectral efficiency. These results arise from the fact that the set partitions in the selected OFDM-FSPM codebook exhibit lower rank than the set partitions in the selected OFDM-OFSPM codebook. Moreover, the codewords related to set partitions in OFDM-OFSPM are capable of preserving the same minimum rank property as the codewords related to the permutations in the MM-OFDM-IM scheme, although the number of index bits for the former increases.

VI. Conclusion

In this paper, we proposed the concept and set partition modulation and described several variants of the technique. We developed the method in the context of OFDM. Moreover, we analyzed the performance of the new techniques in terms of their data rates and BER characteristics, and we compared these against appropriate benchmarks. Through computer simulations and theoretical calculations, we showed that the proposed SPM schemes can provide significant improvements compared to conventional OFDM, OFDM-IM, DM-OFDM-IM, and MM-OFDM-IM. We limited our study to the design and practical model for the SPM schemes; however, we aim to include interesting open problems – such as low-complexity detection, more efficient codebook selection algorithms, and the application of diversity techniques – in our future studies.

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