CONCISE PRESENTATIONS OF DIRECT PRODUCTS

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Abstract. Direct powers of perfect groups admit more concise presentations than one might naively suppose. If $H_1(G,\mathbb{Z}) = H_2(G,\mathbb{Z}) = 0$, then G^n has a presentation with $O(\log n)$ generators and $O(\log n)^3$ relators. If, in addition, there is an element $g \in G$ that has infinite order in every non-trivial quotient of G, then G^n has a presentation with $d(G)+1$ generators and $O(\log n)$ relators. The bounds that we obtain on the deficiency of G^n are not monotone in n; this points to potential counterexamples for the Relation Gap Problem.

1. Introduction

If two groups are presented as $A = \langle X \mid R\rangle$ and $B = \langle Y \mid S\rangle$, then their direct product is given by the presentation with generators $X \sqcup Y$ and relators R, S and $\{[x,y] : x \in X,\, y \in Y\}$. Similarly, if $A_i = \langle X_i \mid R_i\rangle$ with $|X_i| = k_i$ and $|R_i| = l_i$, then the obvious presentation of $A_1 \times \cdots \times A_n$ has $\sum k_i$ generators and $\sum l_i + \sum_{i<j} k_i k_j$ relators. In particular, the direct product A^n of n copies of $A = \langle X \mid R\rangle$ with $|X| = k$ and $|R| = l$ has a presentation with kn generators and $nl + k^2 n(n-1)/2$ relators. In the absence of further hypotheses, one cannot do better than these naive bounds. For example, one cannot generate $\mathbb{Z}^n$ with fewer than n generators, and the number of relators needed to present $\mathbb{Z}^n$ is at least the rank of $H_2(\mathbb{Z}^n,\mathbb{Z}) \cong \mathbb{Z}^n \wedge \mathbb{Z}^n$, which is $n(n-1)/2$. But when $H_1(G,\mathbb{Z})$ and $H_2(G,\mathbb{Z})$ vanish, one can construct much more concise presentations of G^n – that is the main theme of this note.

We shall see that, in addition to the vanishing of homology, the existence of finite quotients of G plays a key role in determining how concise a presentation of G^n can be. The various possibilities are summarised in the following theorem, in which we use the standard notation $d(\Gamma)$ for the minimal number of generators of Γ and we define $\rho(\Gamma)$ to be the minimum number of relators in any finite presentation of Γ. All of the results concerning the growth of $d(G^n)$ are taken from [17]; they draw on earlier results of Hall [8], Wiegold [13, 14, 15] and others. The estimates on $\rho(G^n)$ are new (or trivial).

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We use the standard notation $f(n) = \Theta(g(n))$ for functions that are bounded above and below by positive multiples of $g(n)$, and for brevity we write H_iG in place of $H_i(G, \mathbb{Z})$. Throughout, G^n denotes the direct product of n copies of G.

Theorem 1.1. *Let G be a finitely presented group.*
(1) *If $H_1G \neq 0$, then $d(G^n) = \Theta(n)$ and $\rho(G^n) = \Theta(n^2)$.*
(2) *If $H_1G = 0$ and $H_2G \neq 0$, then $d(G^n) = O(\log n)$ and $\rho(G^n) = \Theta(n)$.*
(3) *If $H_1G = H_2G = 0$, then $d(G^n) = O(\log n)$ and $\rho(G^n) = O(\log n)^3$.*
(4) *If $H_1G = H_2G = 0$ and G has a non-trivial finite quotient, then $d(G^n) = \Theta(\log n)$ and there are constants c_0, c_1 such that $c_0 \log n \leq \rho(G^n) \leq c_1(\log n)^3$.*
(5) *If $H_1G = H_2G = 0$ and there is an element $g \in G$ that has infinite order in every non-trivial quotient of G, then $d(G^n) \leq d(G) + 1$ for all n, and $\rho(G^n) = O(\log n)$.*

In all cases, the upper bounds on $d(G^n)$ and $\rho(G^n)$ can be satisfied simultaneously.

I see no reason to expect that $\rho(G^n)$ is a monotone function of n for all finitely presented perfect groups G, and this is intriguing in the context of the celebrated *Relation Gap Problem* [9]. Recall that the *deficiency* of a finite group presentation $\langle A \mid R \rangle$ is $|R| - |A|$, and the deficiency $\mathrm{def}(G)$ of a group G is defined[1] to be the least deficiency among all finite presentations of G. Our constructions suggest that the following problem might have a positive answer. If it does, then Γ^m would be a counterexample to the Relation Gap Problem: see remark 2.8 for an explanation and variations.

Problem 1.2. *Does there exist a finitely presented perfect group Γ and a positive integer m such that $\mathrm{def}(\Gamma^m) > \mathrm{def}(\Gamma^{m+1})$ or $\rho(\Gamma^m) > \rho(\Gamma^{m+1})$?*

2. Proofs

We shall need some basic facts about universal central extensions of groups.

A *central extension* of a group G is a group $\widetilde{G}$ equipped with an epimorphism $\pi : \widetilde{G} \to G$ whose kernel is central in $\widetilde{G}$. Such an extension is *universal* if given any other central extension $\pi' : E \to G$ of G, there is a unique homomorphism $f : \widetilde{G} \to E$ such that $\pi' \circ f = \pi$. The standard reference for this material is [10] pp. 43–47. The properties that we need here are these: G has a universal central extension $\widetilde{G}$ if (and only if) $H_1(G, \mathbb{Z}) = 0$; there is a short exact sequence

$$1 \to H_2(G, \mathbb{Z}) \to \widetilde{G} \to G \to 1 ;$$

and if G has no non-trivial finite quotients then neither does $\widetilde{G}$.

The following result is Proposition 3.5 of [2].

[1]there are two conventions in the literature: many authors take this definition to be $-\mathrm{def}(G)$.

Lemma 2.1. *Let $G = \langle X \mid R\rangle$ be a perfect group, let F be the free group on X and for each $x \in X$ let $c_x \in [F, F]$ be a word such that $x = c_x$ in G. Then the following is a presentation of the universal central extension of G:*

$$\tilde{G} = \langle X \mid x^{-1}c_x,\ [r, x]\ (\forall r \in R,\ x \in X)\rangle, \tag{2.1}$$

and the identity map $X \to X$ extends uniquely to an epimorphism $\tilde{G} \to G$ with kernel isomorphic to $H_2(G, \mathbb{Z})$.

Corollary 2.2. *If $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$, then (2.1) is a presentation of G.*

It will be convenient to use functional notation for words. Thus, given a word u in the symbols $x_1^{\pm 1}, \dots, x_k^{\pm 1}$, we write $u(\underline{x})$ to emphasize the underlying alphabet and we write $u(\underline{y})$ for the word obtained by replacing each occurrence of each x_i with y_i, where $y_1^{\pm 1}, \dots, y_k^{\pm 1}$ is a second (ordered) alphabet.

Proposition 2.3. *Let $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle$, let F be the free group on the x_i, suppose that $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$, and for each x_i fix $c_i(\underline{x}) \in [F, F]$ such that $x_i = c_i(\underline{x})$ in G. Then the following is a presentation of $G \times G$:*

$$\langle x_1, \dots, x_k, y_1, \dots, y_k \mid r_1, \dots, r_l,\ y_i^{-1}c_i(\underline{y}),\ [x_iy_i^{-1}, y_j],\ 1 \le i, j \le k\rangle. \tag{2.2}$$

Proof. First observe that the last family of relations can be written as $x_i^{-1}y_jx_i = y_i^{-1}y_jy_i$, from which it follows that $x_i^{-1}ux_i = y_i^{-1}uy_i$ for all words u in the free group on $\{y_1, \dots, y_k\}$ and each $i = 1, \dots, k$. Therefore, in the group presented, the transcription $r_j(\underline{y})$ of each relation $r_j(\underline{x})$ is central in the subgroup $G_1 := \langle y_1, \dots, y_k\rangle$, because $y_i = y_i^{r_j(\underline{x})} = y_i^{r_j(\underline{y})}$. Thus G_1 (which is clearly normal) satisfies the relations that were used in Lemma 2.1 to define the universal central extension $\widetilde{G}$. And $\widetilde{G} = G$, because $H_2(G, \mathbb{Z}) = 0$.

At this stage we know that the group given by presentation (2.2) has the form $G_1 \rtimes G_2$, with $G_1 \cong G_2 \cong G$, where G_1 is the subgroup generated by the y_i and G_2 is the subgroup generated by the x_i. The action $\phi : G_2 \to \mathrm{Aut}(G_1)$ defining the semidirect product is by inner automorphisms, $x_i \mapsto \mathrm{ad}_{y_i}$. Because this action factors $G_2 \to G_1 \to \mathrm{Inn}(G_1)$, we have $G_1 \rtimes G_2 \cong G_1 \times G_2$; indeed an isomorphism $\phi : G_1 \times G_2 \to G_1 \rtimes G_2$ is given by $\phi(y_i) = y_i$ and $\phi(x_i) = y_i^{-1}x_i$. ☐

At first blush, this proposition seems to gain us little or nothing compared to the naive presentation of $G \times G$: we have traded the l obvious relations of G_1 for the k relations $y_i^{-1}c_i(\underline{y})$. The real benefit comes when we iterate the construction and use the fact that the number of generators that G^n requires grows strikingly slowly (an old observation of Philip Hall [8]). To exploit this we need:

Lemma 2.4. *Let $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle$ and suppose $H_1G = H_2G = 0$. If G^N requires at most k generators and $2^n \le N$, then G^{2^n} has a presentation with k generators and $n(k^2 + 2k) + l$ relations.*

Proof. As in the previous proof, we construct a presentation of G^2 with $2k$ generators $b_1, \ldots, b_{2k}$ and $k^2 + k + l$ relations. We then make Tietze moves to add a new generating set $a_1, \ldots, a_k$, together with k relations expressing the a_i as words in the generators b_i. There are words u_i in the generators a_j such that $b_i = u_i$ in $G \times G$. We make further Tietze moves, removing the generators b_i and replacing each occurrence of b_i in the relators by u_i. Thus we obtain a presentation of $G \times G$ with k generators and $k^2 + 2k + l$ relators.

Repeating the argument with $G \times G$ in place of G, we obtain a presentation for G^4 with k generators and $2(k^2 + 2k) + l$ relators. And continuing in this manner (provided that we stay in the range where G^{2^n} needs only k generators), we obtain a presentation for G^{2^n} with k generators and $n(k^2 + 2k) + l$ relators. □

Corollary 2.5. *If G and N are as in the lemma and $m \leq N/2$, then G^m has a presentation with k generators and $(k^2 + 2k)(\log_2 m + 1) + l + k$ relators.*

Proof. Let n be the least integer such that $m \leq 2^n$ and write $G^{2^n} = G^m \times G^{2^n - m}$. The lemma tells us that G^{2^n} has a presentation with k generators and $n(k^2+2k)+l$ relators. Moreover, as $2^n - m < N$, the second factor in the given decomposition is a k-generator group, and can therefore be killed by the addition of at most k relations. To complete the proof, note that $n - 1 < \log_2 m$. □

It is an open question as to whether every finitely generated perfect group is the normal closure of one element. If it is, then the k relations added to kill $G^{2^n - m}$ in the above proof could be replaced by a single relation.

We shall need the following result of Wiegold and Wilson [17]; the proof presented here is new but has much in common with the original.

Proposition 2.6. *Let G be a perfect group. If $d(G) = r$, then $d(G^m) \leq r(1 + \lceil \log_2(m+1) \rceil)$.*

Proof. Let $M = \lceil \log_2(m+1) \rceil$, the least integer with $m < 2^M$. The proof uses binary expansions $j = \sum_{i=0}^{M-1} \varepsilon_i(j) 2^i$ of integers $j = 1, \ldots, m$. Given a generating set $\{a_1, \ldots, a_k\}$ for G, for $i = 0, \ldots, M-1$ we define

$$a_{r,i} = (a_r^{\varepsilon_i(1)}, a_r^{\varepsilon_i(2)}, \ldots, a_r^{\varepsilon_i(m)}).$$

For each pair of integers $1 \leq j < j' \leq m$, there is some i such that $\varepsilon_i(j) \neq \varepsilon_i(j')$, and for that i we have $p_{j,j'}(a_{r,i}) \in \{(a_r, 1), (1, a_r)\}$, where $p_{jj'} : G^m \to G \times G$ is the coordinate projection to the j and j' factors. The image under $p_{jj'}$ of the diagonal element $\alpha_r := (a_r, \ldots, a_r)$ is (a_r, a_r). Thus the restriction of $p_{jj'}$ to the subgroup $S < G^n$ generated by the set $\{\alpha_r,\ a_{r,i} \mid r = 1, \ldots, k;\ i = 0, \ldots, M-1\}$ is surjective. It follows that S contains the $(m-1)$-st term of the lower central series of G^m (see [4] p.643). But G^m is perfect, so each term of the lower central series is the whole group, and therefore $S = G^m$. □

Theorem 2.7. *If G is a finitely presented group with $H_1G = H_2G = 0$, then G^m has a finite presentation with at most $O(\log m)$ generators and $O(\log m)^3$ relators.*

Proof. The preceding proposition shows that $d(G^m) = O(\log m)$. We fix a constant k so that G can be generated by k elements and G^m can be generated by $k\lceil\log_2 m\rceil$ elements, for all positive integers m. Suppose $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l\rangle$. Since G^2 only needs k generators, as in Lemma 2.4 we obtain a presentation of G^2 with k generators and k^2+2k+l relators. From Proposition 2.3 we then get a presentation of $G^4 = G^2 \times G^2$ with $2k$ generators and $k^2+k+(k^2+2k+l) = 2k^2+3k+l$ relators. Applying Proposition 2.3 again we get a presentation of G^8 with $4k$ generators and $(2k)^2 + 2k + (2k^2 + 3k + l) = 6k^2 + 5k + l$ relators. Since G^8 only requires $3k$ generators, as in the proof of Lemma 2.4 we can convert this to a presentation with $3k$ generators and $6k^2 + 8k + l$ relators.

Repeating this argument, we obtain a presentation of G^{16} with $6k$ generators and $(3k)^2 + 3k + (6k^2 + 8k + l) = 15k^2 + 11k + l$ relators, which we convert to one with $4k$ generators and $15k^2 + 15k + l$ relators. And, proceeding by induction, we get a presentation of G^{2^n} with nk generators and $\sigma_n k^2 + \tau_n k + l$ relators, where $\sigma_n - 1 = n(n-1)(2n-1)/6$ is the sum of squares up to $(n-1)^2$ and $\tau_n = n^2 - 1$.

Given m, we let $n = \lceil\log_2 m\rceil$, write $G^{2^n} = G^m \times G^{2^n-m}$, take the presentation of G^{2^n} constructed above and kill the factor G^{2^n-m} by adding relations to kill a generating set of cardinality $k\lceil\log_2(2^n - m)\rceil$, which is at most $k(n-1)$. Thus we obtain a presentation of G^m with $kn = O(\log m)$ generators and $\sigma_n k^2 + \tau_n k + l + k(n-1) = O(\log m)^3$ relators. □

Proof of Theorem 1.1 All of the results that we need concerning the growth of $d(G^n)$ can be found in [17]; they draw on earlier results of Hall [8], Wiegold [13, 14, 15] and others. Thus we focus on the estimates for $\rho(G^n)$.

A simple induction using the Künneth formula shows that if $H_1G \neq 0$ then the number of generators needed for H_2G^n is at least $n(n-1)/2$, so one needs at least this number of relations to present G^n. The complementary upper bound is provided by the naive construction in the first paragraph of the Introduction. This proves (1).

If G is perfect, then by the Künneth formula H_2G^n is a direct sum of n copies of H_2G, and therefore $d(H_2G^n)$ grows linearly if $H_2G \neq 0$. This provides the lower bound for (2). To establish a complementary upper bound, we consider the universal central extension $\widetilde{G}$. Theorem 2.7 tells us that $\widetilde{G}^n$ has a presentation with at most $O(\log_2 n)$ generators and $O(\log n)^3$ relations. The kernel of $\widetilde{G} \to G$ is isomorphic to H_2G, so we need only add a further $n\, d(H_2G)$ relations to pass from $\widetilde{G}^n$ to the quotient G^n.

(3) is Theorem 2.7. The bounds on the number of relations in (4) follow from (3) and the simple observation that since $H_1 G^n = 0$, the number of relators in any presentation is at least as great as the number of generators.

If G is perfect and $g \in G$ has infinite order in every non-trivial quotient of G, then G^n is generated by the diagonal copy of G together with $(g, g^2, \ldots, g^n)$, by Theorem 4.4 of [17]; hence $d(G^n) \leq d(G) + 1$, as asserted in (5). The required bound on $\rho(G^m)$ is a special case of Corollary 2.5. □

Relation Gap Problem.

Remark 2.8. If one expresses a finitely presented group G as a quotient of a free group $G \cong F/R$, then the action of F by conjugation on $M = R/[R,R]$ makes M a $\mathbb{Z}F$-module (and a $\mathbb{Z}G$-module). It is obvious that this module requires at most $d_F(R)$ generators, where $d_F(R)$ is the least number of elements (relations of G) that one needs to generate R as a normal subgroup of F. Despite much effort, there is no example known where M is proved to require fewer than $d_F(R)$ generators – the putative difference is the *relation gap*.

An elementary calculation shows that if $N < G$ is normal and perfect, then the relation module for $F \to G/N$ requires no more generators than M does, but one suspects that in some cases G/N is finitely presented and requires more relations than G. For example, if one could prove that there is a finitely presented perfect group Γ such that $\mathrm{def}(\Gamma^n) > \mathrm{def}(\Gamma^m)$ for some $n < m$, then one could take a finite presentation realising the deficiency of Γ^m and add relations to kill a direct factor Γ^{m-n}; the resulting presentation of Γ^n would have a relation gap of at least $\mathrm{def}(\Gamma^n) - \mathrm{def}(\Gamma^m)$.

Similarly, if $\rho(\Gamma^n) > \rho(\Gamma^m)$ for some $n < m$, then by taking a presentation of Γ^m with $\rho(\Gamma_m)$ relators and passing to Γ^n by killing a direct factor Γ^{m-n}, we would obtain a presentation with a relation gap. More generally, it would suffice to prove that a specific map $F \to \Gamma^n$ from a finitely generated free group factored as $F \to \Gamma^m \to \Gamma^n$, where the second map is the quotient by a direct factor and the kernel of the first map requires fewer normal generators than the composite. The special role that powers of the form G^{2^r} play in the proofs of this section is intriguing in this regard.

3. Examples

3.1. **Profinitely trivial examples.** In [3] Fritz Grunewald and I constructed a family of infinite super-perfect groups B_p that have no non-trivial finite quotients. The presentation given there is

$$B_p = \langle a, b, \alpha, \beta \mid ba^p b^{-1} = a^{p+1},\ \beta\alpha^p\beta^{-1} = \alpha^{p+1},\ [bab^{-1}, a]\beta^{-1},\ [\beta\alpha\beta^{-1}, \alpha]b^{-1} \rangle.$$

A 3-generator, 3-relator presentation of B_p can be obtained from this by a simple Tietze move removing the generator β and the third relation, replacing the occurrences of β in the second and fourth relations by the word $[bab^{-1}, a]$.

Lemma 3.1. *Let Q be a quotient of $H = \langle a, b \mid ba^pb^{-1} = a^{p+1}\rangle$. If the image of a in Q has finite order, then the image of $[bab^{-1}, a]$ is trivial.*

Proof. If the image $\overline{a}$ of a has finite order, then the images of a^p and a^{p+1} in Q must have the same order, since they are conjugate. But the order of $\overline{a}^r$ is m/c, where m is the order of $\overline{a}$ and $c = (m, r)$ is the highest common factor. Since p and $p+1$ are coprime, it follows that $\overline{a}^p$ generates $A = \langle\overline{a}\rangle$ and the image of b conjugates $\overline{a}$ to a power of $\overline{a}$. In particular, the image of $[bab^{-1}, a]$ in Q is trivial. □

We need the following strengthening of the fact that B_p has no non-trivial finite quotients.

Proposition 3.2. *$a \in B_p$ has infinite order in every non-trivial quotient of B_p.*

Proof. If the image of a has finite order in a quotient Q, then the image of $[bab^{-1}, a]$ is trivial, by the lemma. The relations $\beta = [bab^{-1}, a]$ and $\beta\alpha^p\beta^{-1} = \alpha^{p+1}$ then force β and α to have trivial image in Q, whence $b = [\beta\alpha\beta^{-1}, \alpha]$ does too. So $Q = 1$. □

Theorem 3.3. *For all integers p, m, the direct product of m copies of B_p has a presentation with at most 4 generators and $24\lceil\log_2 m\rceil - 1$ relations.*

Proof. By Theorem 1.1(5) (which is from [17]) or Remark 3.4(1), we know that B_p^n requires at most 4 generators. To estimate the number of relations needed, first, as in Proposition 2.3, we present $B_p \times B_p$ with 6 generators and 15 relations. Then, as in the proof of Lemma 2.4, we reduce this to a presentation with 4 generators and 19 relations. Continuing the argument of Lemma 2.4, we get a 4-generator presentation of B_p^4 with $16+4+19+4 = 43$ relations, then a 4-generator presentation of B_p^8 with $16 + 4 + 43 + 4 = 67$ relations, a 4-generator presentation of B_p^{16} with $16 + 4 + 67 + 4 = 91$ relations, and a 4-generator presentation of $B_p^{2^n}$ with $24n - 5$ relations. As in Lemma 2.4, we conclude that B_p^m has a 4-generator presentation with at most $24\lceil\log_2 m\rceil - 1$ relations. □

Remarks 3.4. (1) I do not know if the number of relations needed to present B_p^m is $\Omega(\log m)$.

(2) In [1], Baumslag and Miller constructed a 4-generator finitely presented group G_p that admits a surjection $G_p \to G_p \times G_p$. The group B_p is a quotient of G_p, and therefore B_p^n is a quotient of G_p for all positive integers n.

3.2. **Infinite simple groups.** The Burger-Mozes groups are infinite simple groups that arise as the fundamental groups of compact non-positively squared 2-complexes [5]. Such a complex X is a classifying space for its fundamental group $\Gamma = \pi_1 X$, so $H_2\Gamma = H_2X$. These complexes have many more 2-cells than 1-cells, so H_2X is a free-abelian group of non-zero rank. By combining parts (2) and (5) of Theorem 1.1, we see that Γ^n has a finite presentation with at most $d(\Gamma)+1$ generators but the number of relations needed to present Γ^n grows linearly.

Rattaggi [12] refined the original construction of Burger and Mozes to produce examples with relatively small presentations. In particular he constructed an example with 3 generators and 62 relations.

Richard Thompson's group T provides a further example of a 3-generator infinite simple group [11] (see [6], for example). Ghys and Sergiescu [7] proved that $H_2(T,\mathbb{Z}) \neq 0$, so again T^n needs at most 4 generators but the number of relations required to present T^n grows linearly with n.

3.3. **Finite groups.** Super-perfect finite groups are covered by Theorem 1.1(4). It would be particularly interesting to improve the estimate $\rho(G^n) = O(\log n)^3$ in this case, where one has so much more structure.

To close, we follow our construction in the case of the binary icosahedral group $\widetilde{A}_5 \cong \mathrm{SL}(2,5)$. Since it is the universal central extension of A_5, we have $d(\widetilde{A}_5^n) = d(A_5^n)$. Famously, Philip Hall [8] calculated the range of n in which $d(A_5^n) = 2, 3$. By following our argument in this case we get, for example, that $\widetilde{A}_5^{16}$ has a 2-generator presentation with 36 relators, while $\widetilde{A}_5^{1024}$ has a 3-generator presentation with 118 relators.

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