Classes of $\mathscr{C}(\boldsymbol{K})$ spaces with few operators

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To my father

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Abstract

We investigate properties of Koszmider spaces. We show that if K and L are compact Hausdorff spaces with no isolated points, K is Koszmider and $\mathscr{C}(K)$ is isomorphic to $\mathscr{C}(L)$, then K and L are homeomorphic and, in particular, L is also Koszmider. We also analyse topological properties of Koszmider spaces and show that a connected Koszmider space is strongly rigid.

In addition to Koszmider spaces, we introduce the notion of weakly Koszmider spaces. Having established an alternative characterisation thereof, we show that, while it is evident that every Koszmider space is weakly Koszmider, the reverse implication does not hold. We also prove that if $\mathscr{C}(K)$ and $\mathscr{C}(L)$ are isomorphic and K is weakly Koszmider, then so is L. However, if K is Koszmider, there always exists a non-Koszmider space L such that $\mathscr{C}(K)$ and $\mathscr{C}(L)$ are isomorphic.

In the second part of the thesis we present two separable Koszmider spaces the construction of which does not use any set-theoretical assumptions except for the usual (ZFC) axioms. The first space is zero-dimensional, being the Stone space of a Boolean algebra. The second construction results in a separable connected Koszmider space.

Contents

Chapter 1

Introduction

1.1 Historical background

It may be argued that one of the virtues of Banach spaces is their rich operator structure. After all, Banach spaces are sometimes regarded merely as domains for linear operators.

Let us, however, take the opposite perspective and ask the following question: how small can the space $\mathscr{L}^X$ of (bounded linear) operators on a Banach space X be?

As an illustration, Shelah [She78] constructed a nonseparable Banach space X such that every operator on X can be expressed in the form $\lambda \mathrm{I} + S$ where $\lambda \in \mathbb{R}$ and S has separable range. The original construction assumed the Diamond Axiom ($\Diamond$) but ten years later Shelah and Steprāns [SS88] managed to remove it and provided a (ZFC) construction of a space with the above property. Later, Wark [War01] obtained a reflexive space with the same property.

Spaces with few operators are closely associated with the names of Gowers and Maurey and their paper [GM93] in which the authors obtain a space X_{gm} with the following properties:

(a) (see [Lin70]) X_{gm} is indecomposable, that is, it cannot be expressed as a direct sum of two of its infinite-dimensional subspaces. In fact, X_{gm} is *hereditarily* indecomposable, that is, all of its subspaces are indecomposable,

(b) as a corollary to (a), every operator on X_{gm} can be expressed in a form $\lambda \mathrm{I} + S$ with $\lambda \in \mathbb{C}$ and S strictly singular,

(c) (Banach Hyperplane Problem [Ban32]) as a corollary to (b), X_{gm} is not isomorphic to any proper subspace of itself,

(d) (see [BP58]) X_{gm} contains no unconditional basic sequence.

It is worth noting that, while the first known example of a space with property (c) had been obtained slightly earlier, in [Gow94], the question about existence of a space with any of the remaining properties had been open until [GM93].

Recently, Argyros and Haydon [AH08] constructed a Banach space on which every operator has the form $\lambda \mathrm{I} + K$ with $\lambda \in \mathbb{R}$ and K compact.

Suppose now that instead of looking at general Banach spaces we restrict ourselves to spaces of the type $\mathscr{C}(K)$. Here K is an infinite compact Hausdorff space and $\mathscr{C}(K)$ is the Banach space of real-valued continuous functions on K under the supremum norm. We pose the same question: how small can $\mathscr{L}^{\mathscr{C}(K)}$ be?

Certainly, for any g in $\mathscr{C}(K)$, the multiplication operator $g\mathrm{I}\colon f \mapsto gf$ lies in $\mathscr{L}^{\mathscr{C}(K)}$. In addition, it can be shown (see e.g. [Kos04]) that $\mathscr{L}^{\mathscr{C}(K)}$ must contain an operator T which cannot be expressed in the form $T = g\mathrm{I} + K$, where $g \in \mathscr{C}(K)$ and K is compact.

A natural question now arises: does there exist a space K such that every operator T on $\mathscr{C}(K)$ has the form

$$T = g\mathrm{I} + W, \tag{1.1}$$

where $g \in \mathscr{C}(K)$ and W is weakly compact? As was shown by Piotr Koszmider in [Kos04], the answer turns out to be yes. Following the terminology introduced in [Ple04], we call spaces with this property *Koszmider spaces.*

In his paper [Kos04], Koszmider constructed a space K such that

(i) $\mathscr{C}(K)$ is not isomorphic to any proper subspace or quotient,

(ii) $\mathscr{C}(K)$ is indecomposable (although, since every $\mathscr{C}(K)$ contains a copy of c_0, it is not hereditarily indecomposable),

(iii) (see [Sem71]) as a corollary to (ii), $\mathscr{C}(K)$ is not isomorphic to $\mathscr{C}(L)$ for any totally disconnected L.

Of course, as mentioned above, Gowers and Maurey already solved the Banach Hyperplane and the indecomposability problems in [GM93]. However, Koszmider's space was the first such example of a $\mathscr{C}(K)$ space. Motivation for considering property (iii) stems from

Milutin's Theorem [Mil52] which says that $\mathscr{C}(K)$ is isomorphic to $\mathscr{C}(\{0,1\}^\omega)$ whenever K is uncountable and metrisable.

Koszmider showed that if K is a connected space such that every operator on $\mathscr{C}(K)$ is of the form $g\mathrm{I} + W$, as in (1.1), the resulting space $\mathscr{C}(K)$ satisfies (i)–(iii). Assuming the Continuum Hypothesis (CH), he also produced an example of such a space. In the same paper he provided another construction, performed entirely within (ZFC), of a space K satisfying properties (i)–(iii). This K, however, is a representative of another, bigger, class of spaces with few operators, namely, K is the *weakly Koszmider* space. Soon afterwards, Plebanek [Ple04] obtained a (ZFC) construction of a connected Koszmider space.

1.2 Overview of the thesis

In this thesis we investigate further properties of $\mathscr{C}(K)$ spaces with few operators. In particular, we try to establish a connection between the operator structure of $\mathscr{L}^{\mathscr{C}(K)}$ and the topological structure of K. It turns out that the condition of having few operators on $\mathscr{C}(K)$ forces K to have few continuous functions on itself. More precisely, if K is a connected Koszmider space then K is strongly rigid, that is, the only continuous functions on K are the identity and the constants.

We also show that if K and L are Koszmider spaces with no isolated points and $\mathscr{C}(K) \sim \mathscr{C}(L)$ (where "$\sim$" means "Banach-space isomorphic to") then K and L are homeomorphic. In fact, in this setup it is enough to assume that only one of K and L is Koszmider. When put into historical context, this result seems very curious. Recall that K and L are homeomorphic if $\mathscr{C}(K)$ and $\mathscr{C}(L)$ are ring-isomorphic (Gelfand–Kolmogorov, [Sem71, Theorem 7.8.2]) or isometric (Banach–Stone, [Sem71, Theorem 7.8.4]). However, in general, Banach-space isomorphism between $\mathscr{C}(K)$ and $\mathscr{C}(L)$ does not tell us anything about topological relations between K and L. A good illustration of this fact is Milutin's Theorem [Mil52], already mentioned above, which says that $\mathscr{C}(K) \sim \mathscr{C}(\{0,1\}^\omega)$ whenever K is uncountable and metrisable.

In addition to Koszmider spaces, we address the question of what happens if we allow $\mathscr{L}^{\mathscr{C}(K)}$ to contain some operators which are not of the form $g\mathrm{I}+W$. Specifically, we consider the class of centripetal operators (or weak multipliers, as they were called in [Kos04]).

Formally, an operator $T\colon \mathscr{C}(K) \to \mathscr{C}(K)$ is said to be *centripetal* if

$$\lim_{n\to\infty} (Tf_n)(x_n) = 0$$

whenever (f_n) is a bounded disjoint sequence in $\mathscr{C}(K)$ and (x_n) is a sequence in K with $f_n(x_n) = 0$ for each n, and we say that a space K is *weakly Koszmider* if every operator on $\mathscr{C}(K)$ is centripetal. It is clear that every multiplication operator is centripetal and it can be shown that every weakly compact operator is also centripetal. Thus every Koszmider space is weakly Koszmider. We show that the reverse implication does not hold.

It is worth mentioning that weakly Koszmider spaces provide a natural generalisation of Koszmider spaces. It was proved in [Kos04] that an operator T on $\mathscr{C}(K)$ is centripetal if and only if there exist a bounded Borel function $g\colon K \to \mathbb{R}$ and a weakly compact operator $W\colon \mathscr{C}(K)^* \to \mathscr{C}(K)^*$ such that

$$T^* = g\mathrm{I} + W. \tag{1.2}$$

To define the operator $g\mathrm{I}$, recall that, by the Riesz Representation Lemma [Sem71, Theorem 18.4.1], the space $\mathscr{C}(K)^*$ can be identified with the space $\mathscr{M}(K)$ of signed Radon measures on K. Using this representation, we define $g\mathrm{I}\colon \mathscr{M}(K) \to \mathscr{M}(K)$ by setting

$$(g\mathrm{I})(\mu)(f) = \int gf\,\mathrm{d}\mu \quad \forall \mu \in \mathscr{M}(K) \quad \forall f \in \mathscr{C}(K).$$

Note that if g is a continuous function, then $g\mathrm{I}$ is simply the dual of the multiplication operator $f \mapsto gf$ used in the representation (1.1).

As was shown in [Kos04], weakly Koszmider spaces preserve some properties of Koszmider spaces (for example, they have property (i) from the previous section, and, with several additional conditions, they also satisfy (ii) and (iii)). On the other hand, due to softer restrictions on the structure of $\mathscr{L}^{\mathscr{C}(K)}$, they possess extra properties. For example, having established an alternative characterisation of weakly Koszmider spaces, we show that if $\mathscr{C}(K) \sim \mathscr{C}(L)$ and K is weakly Koszmider then so is L. We show that the corresponding result for Koszmider spaces does not hold unless, as above, we assume that K and L do not have isolated points. As a by-product, we construct an example of a weakly Koszmider non-Koszmider space. A similar example was obtained by Fajardo [Faj07].

The results described above constitute the first part of the thesis, Chapter 2 and were published in [Sch08].

The second part of the thesis, Chapters 3 and 4, is devoted to construction of two separable Koszmider spaces. The first, and so far the only known, construction of a separable (and connected) Koszmider space appeared in [Kos04] but, as mentioned in the previous section, assumed (CH). Plebanek's (ZFC) example of a connected space [Ple04] is nonseparable.

By introducing new ingredients to the algorithm from [Kos04], we obtain a separable Koszmider space entirely within (ZFC). Note that if $D = \{d_n : n \in \omega\}$ is a dense subset of K, then the map $T\colon f \mapsto (f(d_n))$ is an embedding of $\mathscr{C}(K)$ into ℓ_∞. Thus we give a positive answer to the question 20 (1188) from [Pea07, p. 575] which asks where there exists in (ZFC) a Koszmider space K such that $\mathscr{C}(K)$ is embedded in ℓ_∞.

In fact, we constructed two separable Koszmider spaces: one is zero-dimensional and another one is connected. Even though the connected example gives a stronger result, we present both constructions here. This is done because the zero-dimensional construction is simpler and served as a motivation for the connected one. The zero-dimensional construction is described in Chapter 3 whilst Chapter 4 deals with the connected case.

A large part of our construction is based on the arguments from [Kos04] and [Ple04]. However, instead of working in $\mathscr{P}(\omega)$, as was done in [Kos04, section 3], or in a measure algebra, as in [Ple04], we construct K as a continuous image of the Gleason space of $\{0,1\}^{2^\omega}$ (in the zero-dimensional case) or $[0,1]^{2^\omega}$ (in the connected case). This gives us the benefit of obtaining separability with no extra effort as both of these spaces are separable.

Another important ingredient of our constructions is property (K) which forces K to be weakly Koszmider (in the zero-dimensional construction we consider property (K$'$) for Boolean algebras and in this case the Stone space $\mathscr{K}(\mathfrak{A})$ of a Boolean algebra $\mathfrak{A}$ with property (K$'$) is weakly Koszmider).

The idea of considering properties (K$'$) and (K) has been inspired by several existing constructions. In his paper [Ple04], Plebanek introduced a so-called property (H) and showed that it forces a space to be weakly Koszmider. The original definition is applicable to any compact Hausdorff space, however, in [Ple04, section 6] Plebanek considered a simplified version of property (H) for zero-dimensional spaces which he called (H$'$) and which is the basis for our property (K$'$). Variations of both of these properties were already used in [Kos04, Theorems 3.1, 5.1] in the same context, although under no specific name.

For completeness we need to mention one more paper. In [Hay81], Haydon gave a definition of the Subsequential Completeness Property (SCP) and showed that if an algebra $\mathfrak{A}$ has the SCP, then $\mathscr{C}(\mathscr{K}(\mathfrak{A}))$ is Grothendieck. Moreover, if, in addition, $\mathfrak{A}$ possesses a certain extra characteristic, then $\mathscr{C}(\mathscr{K}(\mathfrak{A}))$ gives an example of a Grothendieck space with no subspace isomorphic to ℓ_∞. Going back to weakly Koszmider spaces, the property from [Kos04, Theorem 3.1] is pretty much the combination of the SCP and this characteristic, while the property from [Kos04, Theorem 5.1] is its natural generalisation. Our definitions of properties (K′) and (K) are, in turn, respective modifications thereof.

1.3 General notation and terminology

- All Banach spaces are assumed to be infinite-dimensional and over the field $\mathbb{R}$ of real numbers.

- All operators between Banach spaces are assumed to be bounded and linear, and, whenever T is a (bounded linear) operator between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we denote the usual operator norm of T by $\|T\|$, that is,

$$\|T\| = \sup\{\|Tx\|_Y : x \in X \text{ and } \|x\|_X \leqslant 1\}.$$

- All topological spaces are assumed to be infinite and Hausdorff. In addition, a topological space K is assumed to be compact unless it is stated otherwise or it is evident that K cannot be compact (for example, K has the form $L\backslash\{x\}$ with L compact and $x \in L$).

- The topology on $\mathbb{R}$ is assumed to be the usual Euclidean topology.

- Whenever we consider a sequence and the indexing set is not explicitly defined, we assume that it is ω. For example, (f_n) stands for $(f_n)_{n\in\omega}$.

- Let X be a topological space.

  - If $A \subseteq X$, we denote the closure of A by $\overline{A}$ and the interior of A by $\operatorname{int}(A)$.

  - If $A \subseteq X$, we denote the characteristic function of A by χ_A, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X\backslash A. \end{cases}$$

- If $A \subseteq X$ is regarded as a topological space, we assume that A has the subspace topology.
- If $A \subseteq X$ and I is an indexing family, we assume that A^I is a topological space with the product topology.
- We define the identity function on X by I_X, that is,

$$\mathrm{I}_X(x) = x \quad \forall x \in X.$$

- Let X be a topological space and let $f \colon X \to \mathbb{R}$ be a function.
  - We denote the support of f by $\mathrm{supp}(f)$, that is,

$$\mathrm{supp}(f) = \{x \in K : f(x) \neq 0\}.$$

  - We define

$$f^+ = f \vee 0 = \max\{f, 0\}, \quad f^- = -(f \wedge 0) = -\min\{f, 0\}.$$

  Note that $f^+ - f^- = f$ and $f^+ + f^- = |f|$ and that continuity of f implies that both f^+ and f^- are continuous.

- Let X and Y be topological spaces and let $f \colon X \to Y$ be a function.
  - If f is an injection, we write $f \colon X \hookrightarrow Y$ to emphasise this fact.
  - If f is a surjection, we write $f \colon X \twoheadrightarrow Y$ to emphasise this fact.
  - If $A \subseteq X$, we denote the restriction of f to A by $[f]|_A$.

- Let K be a topological space.
  - We define $\mathscr{C}(K)$ to be the Banach space of continuous real-valued functions on K with the supremum norm.
  - We define a partial order $\leqslant$ on $\mathscr{C}(K)$ by saying that if $f, g \in \mathscr{C}(K)$ then

$$f \leqslant g \quad \text{if and only if} \quad f(x) \leqslant g(x) \quad \forall x \in K.$$

  For convenience of notation, if $f \in \mathscr{C}(K)$ and $\varepsilon \in \mathbb{R}$ we say that

$$f \leqslant \varepsilon \quad \text{if and only if} \quad f \leqslant \varepsilon \mathrm{I}_K.$$

- Let (f_n) be a sequence in $\mathscr{C}(K)$. We say that (f_n) is *disjoint* if

$$f_m(x) \cdot f_n(x) = 0 \quad \forall x \in K \quad \forall m \neq n.$$

- We define $\mathscr{M}(K)$ to be the space of signed Radon measures on K. Using the Riesz Representation Lemma [Sem71, Theorem 18.4.1], we will identify $\mathscr{C}(K)^*$ with $\mathscr{M}(K)$.

- Let τ and σ be subsets of ω. We say that
  - τ and σ are *almost disjoint* if $\tau \cap \sigma$ is finite,
  - τ is *almost contained* in σ (denoted by $\tau \subseteq^* \sigma$) if $\tau \backslash \sigma$ is finite.
- Let $\mathcal{A}$ be a subset of an algebra $\mathfrak{A}$. We denote the algebra generated by $\mathcal{A}$ by $\langle \mathcal{A} \rangle$.
- Let $\mathcal{A}$ be a set whose elements are themselves sets. Using the standard set-theoretic notation, we define

$$\bigcup \mathcal{A} = \{X : \exists A \in \mathcal{A} \text{ with } X \in A\}.$$

Chapter 2

Properties of Koszmider and weakly Koszmider spaces

2.1 Introduction

2.1.1 Overview of the chapter

The chapter is organised as follows. Section 2.2 is devoted to proving that if K and L are Koszmider spaces with no isolated points and $\mathscr{C}(K) \sim \mathscr{C}(L)$, then K and L are homeomorphic. It is natural to ask whether the corresponding result holds if we omit the isolated points condition and in order to answer this question, we proceed to section 2.3 which we start with considering centripetal operators and introducing the notion of a weakly Koszmider space (section 2.3.1). In section 2.3.2 we prove that K is weakly Koszmider if and only if the quotient space $\mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(K)}$ is commutative. It follows immediately that if K and L are topological spaces with $\mathscr{C}(K) \sim \mathscr{C}(L)$ then K is weakly Koszmider if and only if so is L.

Using the above criterion, it is possible to obtain further properties of Koszmider spaces and that is done in section 2.3.3. We construct an example of a weakly Koszmider non-Koszmider space and proceed to show that for any Koszmider space K there exists a non-Koszmider space L with $\mathscr{C}(K) \sim \mathscr{C}(L)$. We finish the section with strengthening the result from section 2.2 and show that if K and L are topological spaces without isolated points and $\mathscr{C}(K) \sim \mathscr{C}(L)$, then in order to deduce that K and L are homeomorphic, it is enough to assume that only one of them is Koszmider.

Finally, in section 2.4 we analyse topological properties of Koszmider spaces and show that if K is a connected Koszmider space, the only continuous functions on K are the identity and the constants, that is, K is strongly rigid.

The results of this chapter appeared in [Sch08].

2.1.2 Notation and terminology

- Let X and Y be any sets. We will write

$$\begin{array}{ll} X \sim Y & \text{if } X \text{ and } Y \text{ are isomorphic as Banach spaces,} \\ X \approx Y & \text{if } X \text{ and } Y \text{ are homeomorphic as topological spaces,} \\ X \cong Y & \text{if } X \text{ and } Y \text{ are isomorphic as rings,} \\ X \leqslant Y & \text{if } X \text{ is a subring of } Y, \\ X \trianglelefteq Y & \text{if } X \text{ is an ideal in } Y. \end{array}$$

- Let K be a topological space.
  - We denote the derived set of K by K'.
  - Let $x \in K$. We denote the Dirac measure at x by δ_x, that is,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \qquad \forall A \subseteq K.$$

  - Let $g\colon K \to \mathbb{R}$ be a bounded Borel function. We define the operator $g\mathrm{I}$ on $\mathscr{M}(K)$ by setting

$$(g\mathrm{I})(\mu)(f) = \int gf \mathrm{d}\mu \qquad \forall \mu \in \mathscr{M}(K),\ \forall f \in \mathscr{C}(K).$$

  - Note that when g is a continuous function, $g\mathrm{I}$ is the dual of the multiplication operator $f \mapsto gf$ on $\mathscr{C}(K)$. We shall call this operator $g\mathrm{I}$ again, that is,

$$g\mathrm{I}\colon \mathscr{C}(K) \to \mathscr{C}(K), \quad (g\mathrm{I})(f) = gf.$$

- Let X be a Banach space. We define

$$\begin{aligned} \mathscr{L}^X &= \{T\colon X \to X,\ T \text{ is a (bounded linear) operator}\}, \\ \mathscr{L}^X_{\mathrm{wc}} &= \{T\colon X \to X,\ T \text{ is weakly compact}\}, \end{aligned}$$

  and, if $X = \mathscr{C}(K)$ for some topological space K, we also define

$$\mathscr{L}^X_{\mathrm{M}} = \{g\mathrm{I}\colon X \to X,\ g \in X\}.$$

  We may also denote the above spaces by $\mathscr{L}$, $\mathscr{L}_{\mathrm{wc}}$ and $\mathscr{L}_{\mathrm{M}}$ respectively whenever this notation does not cause ambiguity.

2.2 Properties of Koszmider spaces

We start with a formal definition.

Definition 2.2.1. A space K is said to be *Koszmider* if

$$\mathscr{L}^{\mathscr{C}(K)} = \mathscr{L}_{\mathrm{M}}^{\mathscr{C}(K)} + \mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(K)},$$

that is, for every operator $T\colon \mathscr{C}(K) \to \mathscr{C}(K)$ there exist a continuous $g\colon K \to \mathbb{R}$ and a weakly compact $W\colon \mathscr{C}(K) \to \mathscr{C}(K)$ such that

$$T = g\mathrm{I} + W. \tag{2.1}$$

Examples of Koszmider spaces can be found in [Kos04] and [Ple04]. For general information on $\mathscr{C}(K)$ spaces we refer the reader to [Sem71].

We are interested in the following question: what topological properties of K are preserved under Banach space isomorphisms?

As mentioned in the introduction, while an isometry or a ring isomorphism between $\mathscr{C}(K)$ and $\mathscr{C}(L)$ forces K and L be homeomorphic, a Banach space isomorphism does not in general provide any additional information about common topological properties of K and L.

The picture changes significantly if we assume that K and L have no isolated points and at least one of the spaces is Koszmider. In this case, as we are about to prove, the fact that $\mathscr{C}(K) \sim \mathscr{C}(L)$ implies $K \approx L$.

We start with the following observation. Let K be a topological space. It is well-known (see [HHZ96], for example) that

- $(\mathscr{L}^{\mathscr{C}(K)}, +, \circ)$ is a ring,
- $\mathscr{L}_{\mathrm{M}}^{\mathscr{C}(K)}$ is a subring of $\mathscr{L}^{\mathscr{C}(K)}$,
- $\mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(K)}$ is an ideal in $\mathscr{L}^{\mathscr{C}(K)}$.

Note the following consequence of this fact.

Proposition 2.2.2. *Let K be a topological space. Then*

$$(\mathscr{L}_{\mathrm{M}} + \mathscr{L}_{\mathrm{wc}})/\mathscr{L}_{\mathrm{wc}} \cong \mathscr{L}_{\mathrm{M}}/(\mathscr{L}_{\mathrm{M}} \cap \mathscr{L}_{\mathrm{wc}}) \cong \mathscr{C}(K'). \tag{2.2}$$

In particular, if K is Koszmider, then

$$\mathscr{L}/\mathscr{L}_{\mathrm{wc}} \cong \mathscr{C}(K'). \tag{2.3}$$

For the proof we will use the following version of Claim from the proof of [Kos04, Theorem 2.5], where the corresponding result was proved for an arbitrary function $g\colon K \to \mathbb{R}$.

Lemma 2.2.3. *Let $g \in \mathscr{C}(K)$. Then the operator $g\mathrm{I}\colon \mathscr{C}(K) \to \mathscr{C}(K)$ is weakly compact if and only if for each $\varepsilon > 0$ the set*

$$A_\varepsilon^g = \{x \in K : |g(x)| > \varepsilon\}$$

is finite. Consequently, $g\mathrm{I}$ is weakly compact if and only if each point in $\operatorname{supp}(g)$ is isolated.

For completeness, we include the proof of this result. Since we are only interested in continuous functions, our proof is different from the one in [Kos04] and is based on the following result which will be used throughout the chapter and can be found in [DU77, p.160, Corollary 17].

Theorem 2.2.4. *Let K be a topological space and Y be a Banach space. An operator $T\colon \mathscr{C}(K) \to Y$ is weakly compact if and only if*

$$\lim_{n\to\infty} \|Tf_n\| = 0$$

for every bounded disjoint sequence (f_n) in $\mathscr{C}(K)$.

Proof of Lemma 2.2.3. Suppose that for some $\varepsilon > 0$ there exists an infinite sequence (x_n) in K such that $|g(x_n)| > \varepsilon$ for all n. Passing to subsequences, we may assume that (x_n) is relatively discrete, so that there exist disjoint open U_n with $x_n \in U_n$ for all n. Let (f_n) be a sequence in $\mathscr{C}(K)$ separating (x_n), that is,

$$\|f_n\|_\infty = 1, \quad f_n(x_n) = 1 \quad \text{and} \quad \operatorname{supp}(f_n) \subseteq U_n$$

for each n. Then (f_n) is a bounded disjoint sequence and

$$\|(g\mathrm{I})(f_n)\|_\infty \geqslant |(g\mathrm{I})(f_n)(x_n)| = |g(x_n)f_n(x_n)| > \varepsilon$$

for each n. Theorem 2.2.4 implies that $g\mathrm{I}$ is not weakly compact.

Conversely, suppose that $g\mathrm{I}$ is not weakly compact. Applying Theorem 2.2.4 again and passing to subsequences if needed, we can find a bounded disjoint sequence (f_n) in $\mathscr{C}(K)$ and $\varepsilon > 0$ such that for each n,

$$\|gf_n\|_\infty > \varepsilon,$$

and, in particular, there exists $x_n \in K$ with

$$|g(x_n)f_n(x_n)| > \varepsilon$$

meaning that each x_n lies in $A^g_{\varepsilon/M}$ where M is an upper bound for the set $\{\|f_n\|_\infty : n \in \omega\}$. Disjointness of (f_n) guarantees that all x_n are distinct and so $A^g_{\varepsilon/M}$ is infinite.

For the second part of the lemma, note that

$$\operatorname{supp}(g) = \bigcup_{\varepsilon>0} A^g_\varepsilon.$$

Continuity of g implies that each A^g_ε is open and so if A^g_ε is finite, by the Hausdorff property, it must consist of isolated points. For the converse note that, being compact, K can only have finitely many isolated points. □

Proof of Proposition 2.2.2. The first part of (2.2) is simply the second isomorphism theorem for rings applied to $\mathscr{L}$, $\mathscr{L}_{\mathrm{M}}$ and $\mathscr{L}_{\mathrm{wc}}$.

To prove the second part, define a ring homomorphism

$$\begin{aligned} \Theta\colon \mathscr{L}_{\mathrm{M}} &\to \mathscr{C}(K'), \\ g\mathrm{I} &\mapsto [g]|_{K'}. \end{aligned}$$

Then

$$\begin{aligned} \operatorname{Ker}(\Theta) &= \{g\mathrm{I} \in \mathscr{L}_{\mathrm{M}} : [g]|_{K'} = 0\} \\ &= \{g\mathrm{I} \in \mathscr{L}_{\mathrm{M}} : \operatorname{supp}(g) \subseteq K\backslash K'\} \\ &= \{g\mathrm{I} \in \mathscr{L}_{\mathrm{M}} : \operatorname{supp}(g) \text{ consists of isolated points}\} \\ &= \{g\mathrm{I} \in \mathscr{L}_{\mathrm{M}} : g\mathrm{I} \in \mathscr{L}_{\mathrm{wc}}\} \qquad \text{(by Lemma 2.2.3)} \\ &= \mathscr{L}_{\mathrm{M}} \cap \mathscr{L}_{\mathrm{wc}}. \end{aligned}$$

Furthermore, since K' is a closed subset of K, by the Tietze extension theorem [Wil70, Theorem 15.8], for any $g \in \mathscr{C}(K')$ we can find $\tilde{g} \in \mathscr{C}(K)$ with $[\tilde{g}]|_{K'} = g$ or, equivalently,

$\Theta(\tilde{g}\mathrm{I}) = g$. Thus $\mathrm{Im}(\Theta) = \mathscr{C}(K')$ and the result follows from the first isomorphism theorem for rings. $\square$

Before we proceed, let us mention another consequence of the first isomorphism theorem.

Proposition 2.2.5. *Let X and Y be Banach spaces with $\mathscr{L}^X \sim \mathscr{L}^Y$. Then*

$$\mathscr{L}^X/\mathscr{L}^X_{\mathrm{wc}} \cong \mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}}.$$

Proof. Let $J\colon X \to Y$ be a Banach space isomorphism and define a map

$$\begin{aligned} \Theta\colon \mathscr{L}^X &\to \mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}} \\ T &\mapsto JTJ^{-1} + \mathscr{L}^Y_{\mathrm{wc}}. \end{aligned}$$

Clearly, Θ is a ring homomorphism and $\mathrm{Im}(\Theta) = \mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}}$.

Furthermore, if $T \in \mathscr{L}^X_{\mathrm{wc}}$ then $JTJ^{-1} \in \mathscr{L}^Y_{\mathrm{wc}}$ and so $T \in \mathrm{Ker}(\Theta)$. Conversely, if $T \in \mathrm{Ker}(\Theta)$, then $JTJ^{-1} \in \mathscr{L}^Y_{\mathrm{wc}}$ and so $T = J^{-1}(JTJ^{-1})J \in \mathscr{L}^X_{\mathrm{wc}}$. Thus $\mathrm{Ker}(\Theta) = \mathscr{L}^X_{\mathrm{wc}}$.

The result now follows from the first isomorphism theorem for rings. $\square$

Theorem 2.2.6. *Let K and L be Koszmider spaces such that $\mathscr{C}(K) \sim \mathscr{C}(L)$. Then $K' \approx L'$. In particular, if K and L have no isolated points, then K and L are homeomorphic.*

Proof. Combining Propositions 2.2.2 and 2.2.5,

$$\mathscr{C}(K') \cong \mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}^{\mathscr{C}(K)}_{\mathrm{wc}} \cong \mathscr{L}^{\mathscr{C}(L)}/\mathscr{L}^{\mathscr{C}(L)}_{\mathrm{wc}} \cong \mathscr{C}(L'),$$

and the result follows from the Gelfand–Kolmogorov theorem [Sem71, Theorem 7.8.2]. $\square$

Suppose now that K is Koszmider while L is an arbitrary topological space with $\mathscr{C}(K) \sim \mathscr{C}(L)$. Is it still true that $K' \approx L'$ or perhaps at least that L is also Koszmider?

The answer to both questions, even though negative in general, turns out to be positive if we restrict ourselves to spaces with no isolated points, and this will be analysed in the next section. First, however, we need to introduce some machinery which does not only provide the necessary background but also gives interesting independent results.

2.3 Centripetal operators and weakly Koszmider spaces

2.3.1 Motivation

As mentioned in the introduction, the building block of the constructions in [Kos04] and [Ple04] is the notion of centripetal operators.

Definition 2.3.1. An operator $T\colon \mathscr{C}(K) \to \mathscr{C}(K)$ is said to be *centripetal* if

$$\lim_{n\to\infty} (Tf_n)(x_n) = 0 \tag{2.4}$$

whenever (f_n) is a bounded disjoint sequence in $\mathscr{C}(K)$ and (x_n) is a sequence in K with $f_n(x_n) = 0$ for each n.

This definition was introduced in [Kos04] where such operators were called "weak multipliers". The term "centripetal", however, was used in early drafts of [Kos04] and later appeared in [Ple04]. Both terms relate to exactly the same notion, even though Plebanek's definition may initially seem slightly weaker as he only assumes $x_n \in K\backslash\overline{\operatorname{supp}(f_n)}$ rather than $x_n \in K\backslash\operatorname{supp}(f_n)$.

Lemma 2.3.2. *Let $T\colon \mathscr{C}(K) \to \mathscr{C}(K)$ be an operator. The following statements are equivalent.*

(i) T is centripetal.

(ii) For every bounded disjoint sequence (f_n) in $\mathscr{C}(K)$ and for every sequence (x_n) in K with $x_n \in K\backslash\overline{\operatorname{supp}(f_n)}$ for all n, we have

$$\lim_{n\to\infty} (Tf_n)(x_n) = 0.$$

If, in addition, K is zero-dimensional, the above statements are equivalent to the following.

(iii) For every disjoint sequence (A_n) of clopen subsets of K and for every sequence (x_n) in K with $x_n \notin A_n$ for all n, we have

$$\lim_{n\to\infty} (T\chi_{A_n})(x_n) = 0.$$

Proof. It is trivial that (i) implies (ii). To show the converse, suppose T is noncentripetal. Then we can find $(x_n) \subseteq K$, a bounded disjoint $(f_n) \subseteq \mathscr{C}(K)$ and $\varepsilon > 0$ such that

$$f_n(x_n) = 0 \quad \text{and} \quad |(Tf_n)(x_n)| > \varepsilon$$

for each n. Replacing f_n with a multiple of f_n^+ or f_n^- and passing to subsequences is needed, we may also assume that

$$\operatorname{range}(f_n) \subseteq [0, 1].$$

Fix any n. Note that $\|T\| \neq 0$ as otherwise T would be centripetal. Thus we can define

$$g_n = f_n - \frac{\varepsilon}{4\,\|T\|}\chi_K.$$

Then g_n, g_n^+ and g_n^- all lie in $\mathscr{C}(K)$ and, in addition,

$$\left\|g_n^+\right\|_\infty \leqslant \|f_n\|_\infty \leqslant 1 \quad \text{and} \quad \left\|g_n^-\right\|_\infty \leqslant \frac{\varepsilon}{4\,\|T\|}.$$

Note that

$$\operatorname{supp}(g_n^+) = \{x : f_n(x) > \varepsilon/(4\,\|T\|)\} \subseteq \operatorname{supp}(f_n),$$

and so (g_n^+) is a disjoint sequence. The last inclusion also shows that $x_n \in K \backslash \overline{\operatorname{supp}(g_n^+)}$.

Finally, note that

$$\left|(Tg_n^-)(x_n)\right| \leqslant \left\|(Tg_n^-)\right\|_\infty \leqslant \|T\| \left\|g_n^-\right\|_\infty \leqslant \frac{\varepsilon}{4}$$

while

$$|(Tg_n)(x_n)| \geqslant \left| |(Tf_n)(x_n)| - \left\| T\left(\frac{\varepsilon}{4\,\|T\|}\chi_K\right)\right\|_\infty \right| \geqslant \varepsilon - \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}.$$

Thus

$$\left|(Tg_n^+)(x_n)\right| \geqslant \left||(Tg_n)(x_n)| - \left|(Tg_n^-)(x_n)\right|\right| \geqslant \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

for each n. Consequently, $(Tg_n^+) \not\to 0$ and T does not satisfies the condition in (ii).

Finally, suppose that K is zero-dimensional. Then, by the Stone–Weierstrass theorem [Sem71, Theorem 7.3.8], the algebra generated by the set $\{\chi_A : A \text{ is clopen}\}$ is dense in $\mathscr{C}(K)$. Thus we can replace an arbitrary function f_n in (i) or (ii) with an indicator function χ_{A_n} meaning that (iii) implies (i) and (ii). The reverse implication is trivial. $\square$

An immediate example of centripetal operators is the identity operator or, more generally, any multiplication operator $g\mathrm{I}$ with $g \in \mathscr{C}(K)$. Theorem 2.2.4 provides a less trivial class of examples, namely, weakly compact operators.

The next result was used by both Koszmider [Kos04] and Plebanek [Ple04] in their constructions. Recall that a subspace Y of X is *C^*-embedded* into X if every bounded function in $\mathscr{C}(Y)$ can be extended to a bounded function in $\mathscr{C}(X)$; and a point $x \in X$ is an *open butterfly* if $\{x\} = \overline{V} \cap \overline{W}$ for some open subsets V, W of X.

Theorem 2.3.3 ([Kos04, Theorem 2.7 and Lemma 2.8])**.** *Let K be a topological space. The following statements are equivalent.*

(i) K is Koszmider.

(ii) All operators on $\mathscr{C}(K)$ are centripetal and the space $K\backslash\{x\}$ is C^*-embedded into K for every $x \in K$.

In particular, if all operators on $\mathscr{C}(K)$ are centripetal and K contains no open butterflies, then (ii) holds and K is Koszmider.

The question of determining whether a given space K is Koszmider can now be split into two parts:

(i) Are all operators on $\mathscr{C}(K)$ centripetal?

(ii) Does a certain extra condition (e.g. absence of open butterflies) hold?

However, as was shown in [Kos04], many properties of Koszmider spaces follow from the positive answer to (i) only and do not depend on (ii) at all. This motivates us to introduce another class of spaces.

Definition 2.3.4. A topological space K is said to be *weakly Koszmider* if every operator on $\mathscr{C}(K)$ is centripetal.

Note that Theorem 2.2.4 implies that every Koszmider space is weakly Koszmider. We will see later that the converse is not true in general. Let us summarise several properties of weakly Koszmider spaces which either follow directly from the definition or were proved in [Kos04]. Recall that a Banach space X is said to be *Grothendieck* if every weak*-convergent sequence in X^* converges weakly.

Theorem 2.3.5. *Let K be a weakly Koszmider space. Then*

(i) *$\mathscr{C}(K)$ is Grothendieck; in particular, K does not contain convergent sequences,*

(ii) *if $\phi\colon K \to K$ is continuous and we set $A = \{x \in K : \phi(x) \neq x\}$, then $\phi(A)$ is finite,*

(iii) *an operator T on $\mathscr{C}(K)$ is onto if and only if it is an isomorphism onto its range,*

(iv) *$\mathscr{C}(K)$ is not isomorphic to any of its proper subspaces, nor to any of its proper quotients.*

Proof. (i) The Grothendieck property of $\mathscr{C}(K)$ was shown in [Kos04, Theorem 2.4]. For the second part suppose that (x_n) is an infinite sequence in K converging to x. Then for any f in $\mathscr{C}(K)$ we have

$$\lim_{n\to\infty} |\delta_{x_n}(f) - \delta_x(f)| = \lim_{n\to\infty} |f(x_n) - f(x)| = 0,$$

which means that (δ_{x_n}) is weak*-convergent to δ_x.

Passing to subsequences, we may assume that (x_n) is relatively discrete and there exist open disjoint U_n such that $x_n \in U_n$ for each n. Let (g_n) be a sequence in $\mathscr{C}(K)$ separating (x_n), that is for each n we have

$$\|g_n\|_\infty = 1, \quad g_n(x_n) = 1 \quad \text{and} \quad \operatorname{supp}(g_n) \subseteq U_n.$$

Then

$$\delta_{x_n}(g_n) = 1$$

and, by the Dieudonné–Grothendieck theorem [Die84, VII.14], the set $\{\delta_{x_n} : n \in \omega\}$ is not relatively weakly compact. The Eberlein–Šmulian theorem [Whi67] now implies that $\{\delta_{x_n} : n \in \omega\}$ is not weakly convergent. Thus $\mathscr{C}(K)$ cannot be Grothendieck.

(ii) Suppose that there exists $(x_n) \subseteq A$ with $\{\phi(x_n) : n \in \omega\}$ infinite. We will show that the composition operator $T\colon f \mapsto f \circ \phi$ is noncentripetal.

Indeed, for each n put $y_n = \phi(x_n)$. As before, we may assume that y_n are relatively isolated and there exist disjoint open U_n such that $y_n \in U_n$ for all n. By the Hausdorff property, we may also assume that $x_n \notin U_n$. Let now (f_n) be a sequence in $\mathscr{C}(K)$ separating (y_n), that is,

$$\|f_n\|_\infty = 1, \quad f_n(y_n) = 1 \quad \text{and} \quad \operatorname{supp}(f_n) \subseteq U_n.$$

Then (f_n) is a bounded disjoint sequence and for each n we have

$$f_n(x_n) = 0$$

whilst

$$(Tf_n)(x_n) = (f_n \circ \phi)(x_n) = f_n(y_n) = 1.$$

(iii) This is a weaker version of [Kos04, Theorem 2.3].

(iv) This part is a direct consequence of (iii) and was also mentioned in [Kos04]. Let Y be a subspace of $\mathscr{C}(K)$ such that there exists an isomorphism $J\colon \mathscr{C}(K) \twoheadrightarrow Y \subseteq \mathscr{C}(K)$. Then (iii) implies that $\operatorname{range}(J) = \mathscr{C}(K)$ and thus $Y = \mathscr{C}(K)$.

Similarly, let $Q\colon \mathscr{C}(K) \twoheadrightarrow Y$ be a quotient map and suppose that there exists an isomorphism $J\colon Y \twoheadrightarrow \mathscr{C}(K)$. Then $JQ\colon \mathscr{C}(K) \twoheadrightarrow \mathscr{C}(K)$ is surjective and hence, by the previous part, it must be an isomorphism. This, in turn, forces Q to be an isomorphism as well. □

We now proceed to describe an alternative characterisation of weakly Koszmider spaces which will provide a machinery for obtaining further properties.

2.3.2 Alternative characterisation of weakly Koszmider spaces

It turns out that in order to check whether given space is weakly Koszmider, it is not necessary to go through all operators in $\mathscr{L}^{\mathscr{C}(K)}$ but only through a particular subset thereof.

Definition 2.3.6. Let $S, T \in \mathscr{L}^{\mathscr{C}(K)}$. We define the *commutator* $[S,T]$ of S, T to be the operator

$$[S,T] = ST - TS.$$

Theorem 2.3.7. *Let K be a topological space. The following are equivalent.*

(i) K is weakly Koszmider.

(ii) $\mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(K)}$ is commutative, that is, for any $S, T \in \mathscr{L}^{\mathscr{C}(K)}$, their commutator $[S,T]$ is weakly compact.

Note that the above characterisation is invariant under Banach space isomorphisms. Indeed, if $\mathscr{C}(K) \sim \mathscr{C}(L)$, then, by Proposition 2.2.5, $\mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(K)} \cong \mathscr{L}^{\mathscr{C}(L)}/\mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(L)}$, and thus one of the rings is commutative if and only if another one is. This implies the following result.

Theorem 2.3.8. *Let K and L be topological spaces and suppose that $\mathscr{C}(K) \sim \mathscr{C}(L)$ and K is weakly Koszmider. Then L is also weakly Koszmider.*

The proof of Theorem 2.3.7 requires an alternative characterisation of centripetal operators from [Kos04]. We need to introduce another piece of notation first.

Definition 2.3.9. For every $T \in \mathscr{L}^{\mathscr{C}(K)}$ we define the function $g_T : K \to \mathbb{R}$ by setting

$$g_T(x) = (T^*\delta_x)(\{x\}) \quad \forall x \in K.$$

Theorem 2.3.10 ([Kos04, Theorem 2.2])**.** *An operator $T : \mathscr{C}(K) \to \mathscr{C}(K)$ is centripetal if and only if the set $\{x \in K : \operatorname{osc}(g_T, x) > \varepsilon\}$ is finite for each $\varepsilon > 0$ and $T^* - g_T \mathrm{I}$ is a well-defined weakly compact operator on $\mathscr{M}(K)$.*

We are now ready to prove the main result of the section.

Proof of Theorem 2.3.7. Suppose first that K is weakly Koszmider and let $S, T \in \mathscr{L}^{\mathscr{C}(K)}$. By Theorem 2.3.10, there exist weakly compact $W_S, W_T \in \mathscr{L}^{\mathscr{M}(K)}$ such that

$$S^* = g_S\mathrm{I} + W_S, \quad T^* = g_T\mathrm{I} + W_T.$$

Then

$$\begin{aligned} [S,T]^* &= T^*S^* - S^*T^* = (g_T\mathrm{I} + W_T)(g_S\mathrm{I} + W_S) - (g_S\mathrm{I} + W_S)(g_T\mathrm{I} + W_T) \\ &= (g_T\mathrm{I}g_S\mathrm{I} - g_S\mathrm{I}g_T\mathrm{I}) + (g_T\mathrm{I}W_S + W_Tg_S\mathrm{I} + W_TW_S - g_S\mathrm{I}W_T - W_Sg_T\mathrm{I} - W_SW_T) \\ &= g_T\mathrm{I}W_S + W_Tg_S\mathrm{I} + W_TW_S - g_S\mathrm{I}W_T - W_Sg_T\mathrm{I} - W_SW_T. \end{aligned}$$

Since $\mathscr{L}_{\mathrm{wc}}^{\mathscr{M}(K)}$ is an ideal, it follows that the last operator is weakly compact. Gantmacher's theorem [HHZ96, Theorem 319] now implies that $[S,T] \in \mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(K)}$.

Conversely, suppose that $\mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}_{\mathrm{wc}}^{\mathscr{C}(K)}$ is commutative. Note first that $\mathscr{C}(K)$ is Grothendieck. Indeed, if not, then (see [Sch82]) we can find a disjoint sequence (f_n) of elements in $\mathscr{C}(K)$ of norm 1, such that the space $Y = \overline{\operatorname{span}}\{f_n : n \in \omega\}$ is a copy of c_0 and is complemented in $\mathscr{C}(K)$. Let $P : \mathscr{C}(K) \twoheadrightarrow Y$ be a projection, $\iota : Y \hookrightarrow \mathscr{C}(K)$ be the inclusion map, and consider the operators $S, T : Y \to Y$ which are the continuous linear extensions of the shift-type operators $f_n \mapsto f_{n+1}$ and $f_n \mapsto f_{2n}$ respectively. Then for each n we have

$$\|[\iota SP, \iota TP](f_n)\|_\infty = \|f_{2n+1} - f_{2n+2}\|_\infty = 1,$$

and so, by Theorem 2.2.4, $[\iota SP, \iota TP]$ is not weakly compact.

Suppose now that K is not weakly Koszmider. Then there exist $T \in \mathscr{L}^{\mathscr{C}(K)}$, a bounded disjoint $(f_n) \subseteq \mathscr{C}(K)$, a sequence $(x_n) \subseteq K$ and $\varepsilon > 0$ such that

$$f_n(x_n) = 0 \quad \text{and} \quad |(Tf_n)(x_n)| > \varepsilon \tag{2.5}$$

for each n. Replacing f_n with f_n^+ or f_n^- and passing to subsequences if needed, we may assume that each f_n takes nonnegative values only, and so, in particular, that $f_n^{1/2}$ is well-defined.

Define a sequence of functionals $(\phi_n) \subseteq \mathscr{C}(K)^*$ by setting

$$\phi_n(g) = [g\mathrm{I}, T](f_n^{1/2})(x_n) = g(x_n)(Tf_n^{1/2})(x_n) - (T(gf_n^{1/2}))(x_n) \quad \forall g \in \mathscr{C}(K).$$

Note that $(f_n^{1/2})$ is a bounded disjoint sequence and, by (2.5),

$$\begin{aligned} \left|\phi_n(f_n^{1/2})\right| &= \left|f_n^{1/2}(x_n)(Tf_n^{1/2})(x_n) - (T(f_n^{1/2}f_n^{1/2}))(x_n)\right| \\ &= |T(f_n)(x_n)| \\ &> \varepsilon \end{aligned}$$

for each n. Thus, by the Dieudonné–Grothendieck theorem [Die84, VII.14], $\{\phi_n : n \in \omega\}$ is not relatively weakly compact. By the Eberlein–Šmulian theorem [Whi67], (ϕ_n) is not weakly convergent. But $\mathscr{C}(K)$ is Grothendieck, and so (ϕ_n) is not weak*-convergent. In particular, there exists g in $\mathscr{C}(K)$ with $\phi_n(g) \not\to 0$ as $n \to \infty$. Passing to subsequences if necessary, we can find $\delta > 0$, such that $|\phi_n(g)| > \delta$ for each n which means that

$$\left\|[g\mathrm{I}, T](f_n^{1/2})\right\|_\infty \geqslant \left|[g\mathrm{I}, T](f_n^{1/2})(x_n)\right| = |\phi_n(g)| > \delta,$$

and, by Theorem 2.2.4, $[g\mathrm{I}, T]$ is not weakly compact which is a contradiction. □

2.3.3 Further properties of Koszmider and weakly Koszmider spaces

We start with showing that the classes of Koszmider and weakly Koszmider spaces do not coincide.

Proposition 2.3.11. *There exists a weakly Koszmider non-Koszmider space.*

More precisely, let K be a (weakly) Koszmider space and x_0, x_1 be distinct non-isolated points in K. Form a quotient space $K_{\mathcal{R}}$ by identifying x_0 and x_1 (that is, we define an *equivalence relation $\mathcal{R}$ on K by saying that $x\mathcal{R}y$ if and only if $\{x,y\}=\{x_0,x_1\}$ or $x=y$, and set $K_{\mathcal{R}}=K/\mathcal{R}$). Then $K_{\mathcal{R}}$ is weakly Koszmider non-Koszmider space.*

The proof uses the following auxiliary result.

Lemma 2.3.12. *Let X be a Banach space and suppose that Y is a subspace of X of finite codimension. Then*

$$\mathscr{L}^X/\mathscr{L}^X_{\mathrm{wc}} \cong \mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}}.$$

Proof. Since finite-codimensional subspaces are complemented (see e.g. [HHZ96]), there exists a projection $P\colon X \twoheadrightarrow Y$. Let $\iota\colon Y \hookrightarrow X$ be the inclusion map. Since $[P]|_Y = \mathrm{I}_Y$, it follows that $P\iota = \mathrm{I}_Y$. It also follows that $\mathrm{I}_X - \iota P$ has finite rank meaning that $\overline{(\mathrm{I}_X-\iota P)(B_X)}$ is a closed bounded subset of a finite-dimensional space and hence is (weakly) compact.

For any $T\in\mathscr{L}^X$ we set $\tilde{T}=PT\iota$. Then $\tilde{T}\in\mathscr{L}^Y$ and for any $S,T\in\mathscr{L}^X$ we have

$$\widetilde{S+T}=\tilde{S}+\tilde{T} \quad \text{and} \quad \widetilde{ST}-\tilde{S}\tilde{T} = PS(\mathrm{I}_X-\iota P)T\iota \in \mathscr{L}^Y_{\mathrm{wc}},$$

and so we can define the following ring homomorphism

$$\begin{aligned} \Theta\colon \mathscr{L}^X &\to \mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}}, \\ T &\mapsto \tilde{T}+\mathscr{L}^Y_{\mathrm{wc}}. \end{aligned}$$

Note that if $S\in\mathscr{L}^Y$, then $\iota SP\in\mathscr{L}^X$ and

$$\widetilde{\iota SP} = P\iota SP\iota = \mathrm{I}_Y S \mathrm{I}_Y = S.$$

Thus $\mathrm{Im}(\Theta) = \mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}}$. Furthermore, $\mathrm{Ker}(\Theta)=\mathscr{L}^X_{\mathrm{wc}}$. Indeed, if $T\in\mathrm{Ker}(\Theta)$, then $\tilde{T}\in\mathscr{L}^Y_{\mathrm{wc}}$. Hence the operator $\iota\tilde{T}P=\iota PT\iota P$ is weakly compact. But, by above, so is $(\mathrm{I}_X-\iota P)T\iota P$. Thus

$$T\iota P = (\mathrm{I}_X-\iota P)T\iota P + (\iota P)T\iota P \in \mathscr{L}^X_{\mathrm{wc}}.$$

Similarly, since $T(\mathrm{I}_X-\iota P)\in\mathscr{L}^X_{\mathrm{wc}}$, we have

$$T = T(\mathrm{I}_X-\iota P)+T\iota P\in\mathscr{L}^X_{\mathrm{wc}}.$$

Conversely, of course, if $T\in\mathscr{L}^X_{\mathrm{wc}}$ then $\tilde{T}\in\mathscr{L}^Y_{\mathrm{wc}}$ and so $T\in\mathrm{Ker}(\Theta)$.

The result now follows from the first isomorphism theorem for rings. ◻

Proof of Proposition 2.3.11. Let $\pi_{\mathcal{R}}\colon K \twoheadrightarrow K_{\mathcal{R}}$ be the quotient map and let $g \in \mathscr{C}(K)$ be a function separating x_0 and x_1, that is,

$$\|g\|_\infty = 1, \quad g(x_0) = 0 \quad \text{and} \quad g(x_1) = 1.$$

Consider the following subspace of $\mathscr{C}(K)$

$$Y = \{f \in \mathscr{C}(K) : f(x_0) = f(x_1)\}.$$

Note that every $f \in \mathscr{C}(K)$ can be expressed as

$$f = -(f(x_0) - f(x_1))g + [f + (f(x_0) - f(x_1))g],$$

where

$$[f + (f(x_0) - f(x_1))g](x_0) = [f + (f(x_0) - f(x_1))g](x_1) = f(x_0)$$

meaning that $f + (f(x_0) - f(x_1))g \in Y$. Consequently, the singleton $\{g + Y\}$ forms a basis for $\mathscr{C}(K)/Y$ and so Y has codimension 1 in $\mathscr{C}(K)$. Lemma 2.3.12 implies that

$$\mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}} \cong \mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}^{\mathscr{C}(K)}_{\mathrm{wc}}.$$

Furthermore, note that the map $f \mapsto f\pi_{\mathcal{R}}$ is an isomorphism between $\mathscr{C}(K_{\mathcal{R}})$ and Y. Applying Proposition 2.2.5 and using the previous equation, we get

$$\mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}^{\mathscr{C}(K)}_{\mathrm{wc}} \cong \mathscr{L}^Y/\mathscr{L}^Y_{\mathrm{wc}} \cong \mathscr{L}^{\mathscr{C}(K_{\mathcal{R}})}/\mathscr{L}^{\mathscr{C}(K_{\mathcal{R}})}_{\mathrm{wc}}.$$

Now, K is weakly Koszmider, hence, by Theorem 2.2.4, $\mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}^{\mathscr{C}(K)}_{\mathrm{wc}}$ is commutative. Consequently, $\mathscr{L}^{\mathscr{C}(K_{\mathcal{R}})}/\mathscr{L}^{\mathscr{C}(K_{\mathcal{R}})}_{\mathrm{wc}}$ is also commutative and, again by Theorem 2.2.4, $K_{\mathcal{R}}$ is weakly Koszmider.

To show that $K_{\mathcal{R}}$ is not Koszmider, we define

$$z = \pi_{\mathcal{R}}(x_0).$$

Then $K_{\mathcal{R}}\backslash\{z\}$ is not C^*-embedded into $K_{\mathcal{R}}$. Indeed, consider the function

$$h = \left[g\pi_{\mathcal{R}}^{-1}\right]\big|_{K_{\mathcal{R}}\backslash\{z\}}.$$

Boundedness of g guarantees that h itself is bounded whilst continuity of g and $\left[\pi_{\mathcal{R}}^{-1}\right]\big|_{K_{\mathcal{R}}\backslash\{z\}}$ implies that $h \in \mathscr{C}(K_{\mathcal{R}}\backslash\{z\})$.

However, there does not exist a continuous extension of h onto the whole of $K_{\mathcal{R}}$. Indeed, suppose $\tilde{h}$ were such an extension. Consider any $\varepsilon > 0$. By continuity of $\tilde{h}$ at z there exists an open (in $K_{\mathcal{R}}$) $U_{\mathcal{R}} \ni z$ such that

$$\left|\tilde{h}(z) - \tilde{h}(y_{\mathcal{R}})\right| < \frac{\varepsilon}{2} \quad \forall y_{\mathcal{R}} \in U_{\mathcal{R}}. \tag{2.6}$$

Similarly, by continuity of g at x_0, there exists an open (in K) $V \ni x_0$ such that

$$|g(x_0) - g(y)| < \frac{\varepsilon}{2} \quad \forall y \in V. \tag{2.7}$$

Consider now the set $W = V \cap \pi_{\mathcal{R}}^{-1}(U_{\mathcal{R}})$. Clearly, W is open in K and $x_0 \in W$. Since x_0 is non-isolated and K is Hausdorff, W is infinite and, in particular, we can find a point $y \in W \backslash \{x_0, x_1\}$.

Then, on the one hand, $y \in V$ which means that

$$|g(x_0) - g(y)| < \frac{\varepsilon}{2}.$$

On the other hand, $\pi_{\mathcal{R}}(y) \in U_{\mathcal{R}} \backslash \{z\}$ and hence, by (2.6),

$$\begin{aligned} \frac{\varepsilon}{2} &> \left|\tilde{h}(z) - \tilde{h}(\pi_{\mathcal{R}}(y))\right| \\ &= \left|\tilde{h}(z) - h(\pi_{\mathcal{R}}(y))\right| \\ &= \left|\tilde{h}(z) - g(\pi_{\mathcal{R}}^{-1}(\pi_{\mathcal{R}}(y)))\right| \\ &= \left|\tilde{h}(z) - g(y)\right|. \end{aligned}$$

Thus

$$\left|\tilde{h}(z) - g(x_0)\right| \leqslant \left|\tilde{h}(z) - g(y)\right| + |g(y) - g(x_0)| < \varepsilon.$$

The last expression is true for any $\varepsilon > 0$ which leads to the conclusion that

$$\tilde{h}(z) = g(x_0) = 0.$$

Note that the above argument remains valid if we replace x_0 with x_1 throughout the text. This, however, leads to a contradiction, as this would imply that

$$\tilde{h}(z) = g(x_1) = 1.$$

Thus $K_{\mathcal{R}} \backslash \{z\}$ is not C^*-embedded into $K_{\mathcal{R}}$ and, by Theorem 2.3.3, $K_{\mathcal{R}}$ is not Koszmider.

□

As mentioned in the introduction, another weakly Koszmider non-Koszmider space was independently obtained by Fajardo in his PhD thesis [Faj07]. His example is, in fact, the same as ours, but the proof that the resulting space satisfies the required conditions is different.

Using the construction from Proposition 2.3.11, we may finally answer the questions posed in the end of section 2.2.

Proposition 2.3.13. *Let K be a Koszmider space. There exists a non-Koszmider space L with $\mathscr{C}(K) \sim \mathscr{C}(L)$ and $K' \not\approx L'$.*

Proof. As in Proposition 2.3.11, choose non-isolated points $x_0, x_1 \in K$ and form $K_{\mathcal{R}}$ by identifying x_0 and x_1 into a point z. Then, as we saw, $K_{\mathcal{R}}$ is a weakly Koszmider non-Koszmider space and $\mathscr{C}(K_{\mathcal{R}})$ is isomorphic to a hyperplane of $\mathscr{C}(K)$ meaning that

$$\mathscr{C}(K) \sim \mathscr{C}(K_{\mathcal{R}}) \oplus \mathbb{R}. \tag{2.8}$$

Pick now any point $w \notin K_{\mathcal{R}}$ and form L by adding w to $K_{\mathcal{R}}$ as an isolated point. That is, $L = K_{\mathcal{R}} \cup \{w\}$ and the topology on L is generated by the sets open in $K_{\mathcal{R}}$, and by $\{w\}$. Since $K_{\mathcal{R}} \backslash \{z\}$ is not C^*-embedded into $K_{\mathcal{R}}$, it follows that $L \backslash \{z\}$ is not C^*-embedded into L, and so L is not Koszmider. However, since w is isolated,

$$\mathscr{C}(L) \sim \mathscr{C}(K_{\mathcal{R}}) \oplus \mathbb{R}, \tag{2.9}$$

which, combined with (2.8), gives us that $\mathscr{C}(K) \sim \mathscr{C}(L)$.

Note that since isolated points do not change centripetality of operators or the property of being C^*-embedded, $K' \backslash \{x\}$ is C^*-embedded into K' for each $x \in K'$ while $L' \backslash \{z\}$ is not C^*-embedded into L'. Thus $K' \not\approx L'$. □

Let us finish the section with a positive result. Suppose that K is Koszmider and L is a topological space with $\mathscr{C}(K) \sim \mathscr{C}(L)$. Assume also that K and L have no isolated points.

Then

$$
\begin{aligned}
\mathscr{C}(K) = \mathscr{C}(K') &\cong \mathscr{L}^{\mathscr{C}(K)}/\mathscr{L}^{\mathscr{C}(K)}_{\mathrm{wc}} && \text{(Proposition 2.2.2)}\\
&\cong \mathscr{L}^{\mathscr{C}(L)}/\mathscr{L}^{\mathscr{C}(L)}_{\mathrm{wc}} && \text{(Proposition 2.2.5)}\\
&\geqslant (\mathscr{L}^{\mathscr{C}(L)}_{\mathrm{M}} + \mathscr{L}^{\mathscr{C}(L)}_{\mathrm{wc}})/\mathscr{L}^{\mathscr{C}(L)}_{\mathrm{wc}} && (\mathscr{L}^{\mathscr{C}(L)}_{\mathrm{M}} + \mathscr{L}^{\mathscr{C}(L)}_{\mathrm{wc}} \leqslant \mathscr{L}^{\mathscr{C}(L)})\\
&\cong \mathscr{L}^{\mathscr{C}(L)}_{\mathrm{M}}/(\mathscr{L}^{\mathscr{C}(L)}_{\mathrm{M}} \cap \mathscr{L}^{\mathscr{C}(L)}_{\mathrm{wc}}) && \text{(Proposition 2.2.2)}\\
&\cong \mathscr{C}(L') && \text{(Proposition 2.2.2)}\\
&= \mathscr{C}(L).
\end{aligned}
$$

To summarise, $\mathscr{C}(L)$ is ring isomorphic to a subspace Y of $\mathscr{C}(K)$ and is Banach space isomorphic to $\mathscr{C}(K)$. Part (iii) of Theorem 2.3.5 implies that $Y = \mathscr{C}(K)$, that is, $\mathscr{C}(K) \cong \mathscr{C}(L)$. The Gelfand–Kolmogorov theorem [Sem71, Theorem 7.8.2] now implies that $K \approx L$ and, in particular, L is also Koszmider. Thus we have just proved that following result.

Theorem 2.3.14. *Let K and L be topological spaces with no isolated points and suppose that K is Koszmider and $\mathscr{C}(K) \sim \mathscr{C}(L)$. Then $K \approx L$ and, in particular, L is also Koszmider.*

2.4 Strong rigidity of connected Koszmider spaces

We already mentioned that connected Koszmider spaces play a special role in the theory of Banach spaces as, being indecomposable, they provide the first known class of examples of $\mathscr{C}(K)$ spaces which are not isomorphic to $\mathscr{C}(L)$ for any totally disconnected L. For more details we refer the reader to [Kos04], where Koszmider proves an even stronger result, namely, if K is a weakly Koszmider space and $K\backslash F$ is connected for any finite F, then $\mathscr{C}(K) \not\sim \mathscr{C}(L)$ for any zero-dimensional space L.

It turns out that connected Koszmider spaces are also interesting from a topological point of view because the condition of having few operators on $\mathscr{C}(K)$ forces K to have few continuous functions on itself. To make the last statement more precise we need to recall one more definition.

Definition 2.4.1. Let K be a topological (but not necessarily compact or Hausdorff) space. We say that K is *strongly rigid* if the only continuous non-constant function from K to itself is the identity.

De Groot [dG59] proved that strongly rigid Hausdorff spaces exist and Kannan and Rajagopalan [KR78] showed that under an extra set-theoretic assumption, namely, $(2^{\mathfrak{m}})^+ < 2^{2^{\mathfrak{m}}}$ for every infinite cardinal $\mathfrak{m}$, it is possible to construct a Hausdorff (not necessarily compact) strongly rigid space of an arbitrarily large cardinality. We will show now that connected Koszmider spaces provide another class of examples of strongly rigid spaces. For this we need to establish several intermediate results. The first one is an easy consequence of Theorem 2.3.3.

Proposition 2.4.2. *Let K be a topological space. The following are equivalent.*

(i) K is Koszmider.

(ii) All operators on $\mathscr{C}(K)$ are centripetal and the space $K\backslash F$ is C^*-embedded into K for every finite $F \subseteq K$.

Proof. (ii)⇒(i) follows trivially from Theorem 2.3.3. To show (i)⇒(ii), suppose that K is Koszmider and consider any finite $F = \{x_0, \dots, x_n\} \subseteq K$ and a bounded continuous $h\colon K\backslash F \to [0,1]$.

Let $(U_i)_{0\leqslant i\leqslant n}$ be a sequence of open subsets of K with mutually disjoint closures and such that $U_i \ni x_i$ for each i. We construct a sequence $(g_i)_{0\leqslant i\leqslant n} \subseteq \mathscr{C}(K)$ using the following algorithm.

- For $i \neq n$ we define g_i to be a continuous function separating x_i from $K\backslash U_i$, that is,

$$g_i \in \mathscr{C}(K), \quad \mathrm{range}(g_i) \subseteq [0,1], \quad g_i(x_i) = 1 \quad \text{and} \quad \mathrm{supp}(g_i) \subseteq U_i.$$

- For $i = n$ we define

$$g_n = \chi_K - \sum_{i=0}^{n-1} g_i.$$

This way we have

$$g_i \in \mathscr{C}(K), \quad \mathrm{range}(g_i) \subseteq [0,1], \quad g_i(x_j) = \delta_{ij} \quad \forall i,j \in \{0,\dots,n\},$$
$$\sum_{i=0}^{n} g_i = \chi_K.$$

For each i define now a function $h_i\colon K\backslash\{x_i\} \to \mathbb{R}$ by setting

$$h_i(x) = \begin{cases} h(x)g_i(x) & \text{if } x \in K\backslash F \\ 0 & \text{otherwise} \end{cases}$$

Then h_i is bounded with $\|h_i\|_\infty \leqslant \|h\|_\infty$. In addition, $h_i \in \mathscr{C}(K\backslash\{x_i\})$. Indeed, h_i is clearly continuous on $\operatorname{int}(K\backslash F) = K\backslash F$. Furthermore, consider any $j \in \{0, \dots, n\}\backslash\{i\}$ and $\varepsilon > 0$. If $\|h\|_\infty = 0$, then $h_i \equiv 0$ and so h_i is continuous. Otherwise, since $g_i(x_j) = 0$, by continuity of g_i, there exists an open $V \ni x_j$ with

$$|g_i(y)| < \frac{\varepsilon}{\|h\|_\infty} \quad \forall y \in V.$$

Then

$$|h_i(y)| \leqslant |h(y)g_i(y)| < \|h\|_\infty \frac{\varepsilon}{\|h\|_\infty} = \varepsilon \quad \forall y \in V,$$

and so h_i is continuous at x_j.

To summarise, $h_i\colon K\backslash\{x_i\} \to \mathbb{R}$ is a bounded continuous function. Theorem 2.3.3 implies that h_i has an extension $\tilde{h}_i \in \mathscr{C}(K)$.

Consider now the function

$$\tilde{h} = \sum_{i=0}^{n} \tilde{h}_i.$$

By above, $\tilde{h} \in \mathscr{C}(K)$. Note also that if $x \notin F$, we have

$$\tilde{h}(x) = \sum_{i=0}^{n} \tilde{h}_i(x) = \sum_{i=0}^{n} h(x)g_i(x) = h(x)\sum_{i=0}^{n} g_i(x) = h(x),$$

and so $\tilde{h}$ is a continuous extension of h. □

Corollary 2.4.3. *Let K be a connected Koszmider space and consider any finite subset F of K. Then $K\backslash F$ is also connected.*

Proof. Endow the space $\{0, 1\}$ with the discrete topology and consider any continuous $g\colon K\backslash F \to \{0, 1\}$. By Proposition 2.4.2, there exists $\tilde{g} \in \mathscr{C}(K)$ extending g. Then $\tilde{g}(K)$ is a nonempty connected subset of $\mathbb{R}$ which is also finite, being a union of the finite sets $g(K\backslash F)$ and $\tilde{g}(F)$. This is only possible if $\tilde{g}(K)$ is a singleton. Equivalently, $\tilde{g}$ is a constant map. This, in turn, implies that g is constant and so $K\backslash F$ is connected. □

We are now ready to prove the main result of the section.

Theorem 2.4.4. *Let K be a connected Koszmider space. Then K is strongly rigid.*

Proof. Let $\phi\colon K \to K$ be a continuous non-identity function. Define

$$A = \{x \in K : \phi(x) \neq x\}.$$

Then A is a nonempty open subset of K and hence it must be infinite. It follows from part (ii) of Theorem 2.3.5 that $\phi(A)$ is finite, and so, by Corollary 2.4.3, $K\backslash\phi(A)$ is connected. Two cases are now possible.

Case 1. There exists some $y \in \phi(A) \cap A$.

Finiteness of $\phi(A)$ means that we can find an open U with $U \cap \phi(A) = \{y\}$. Furthermore, since $\phi(x) = x$ for each $x \notin A$, we have $\phi(K) = (K\backslash A) \cup \phi(A)$ and so

$$(U \cap A) \cap \phi(K) = U \cap A \cap \phi(A) = \{y\}$$

meaning that y is an isolated point of $\phi(K)$ which is a nonempty connected subset of $\mathbb{R}$. This means that $\phi(K)$ is a singleton and ϕ is a constant map.

Case 2. $\phi(A) \cap A = \emptyset$.

In this case $\phi(A) \subseteq K\backslash A$ which means that

$$\phi^{-1}(\phi(A)) = \phi(A) \cup A,$$

and so

$$A = \phi^{-1}(\phi(A)) \cap K\backslash\phi(A).$$

Now, $\phi^{-1}(\phi(A))$ is closed in K, being a continuous preimage of a finite subset of a Hausdorff space. Thus A is closed in $K\backslash\phi(A)$. But A is open in K and so it must be open in $K\backslash\phi(A)$. By our assumption, A is nonempty and so connectedness of $K\backslash\phi(A)$ implies that $A = K\backslash\phi(A)$. Thus

$$\phi(K\backslash A) = K\backslash A = \phi(A)$$

which means that $\phi(K) = \phi(A)$ and so $\phi(K)$ is a finite nonempty connected subset of $\mathbb{R}$. Thus $\phi(K)$ is a singleton and ϕ is a constant map. $\square$

Note that the proof also works for any weakly Koszmider space K such that $K\backslash F$ is connected whenever F is finite.

Chapter 3

Construction of a separable zero-dimensional Koszmider space

3.1 Introduction

3.1.1 Description of the construction

As mentioned in the introduction, the only known construction of a separable Koszmider space, described in [Kos04], assumes (CH). In this section we present a space K the construction of which is carried out entirely in (ZFC).

In order to show that K has the required properties, we prove that

(P1) K is separable,

(P2) K is weakly Koszmider,

(P3) K has no open butterflies.

Our space has the form $\mathscr{K}(\mathfrak{A})$, the Stone space of a Boolean algebra $\mathfrak{A}$, and so is zero-dimensional.

A large part of our construction is based on the arguments from [Kos04] and [Ple04]. However, instead of working in $\mathscr{P}(\omega)$, as was done in [Kos04], or in a measure algebra, as in [Ple04], we construct $\mathfrak{A}$ as a subalgebra of $\mathfrak{R}_{\{0,1\}^{2^\omega}}$, the algebra of regular open subsets of $\{0,1\}^{2^\omega}$. This gives us the benefit of obtaining (P1) with no extra effort: since $\{0,1\}^{2^\omega}$ is separable, so is its Gleason space $\mathscr{G}_{\{0,1\}^{2^\omega}} = \mathscr{K}(\mathfrak{R}_{\{0,1\}^{2^\omega}})$ and consequently $\mathscr{K}(\mathfrak{A})$ is itself separable, being a continuous image of a separable space.

In order to obtain (P2), we introduce property (K$'$) and show that the Stone space of an algebra with property (K$'$) is weakly Koszmider.

The definition of property (K$'$) is based on property (H$'$) from [Ple04] and the property described in [Kos04, Theorem 3.1]. The main difference is that, while Koszmider and Plebanek considered pairs $((A_n),(x_n))$ with (A_n) being a disjoint sequence of clopen subsets of K and (x_n) being a sequence in K with $\bigcup A_n \cap \{x_n\} = \emptyset$ (or $\overline{\bigcup A_n} \cap \overline{\{x_n\}} = \emptyset$), we look at pairs of the form $((A_n),(B_n))$ where both (A_n), (B_n) are disjoint sequences of clopen subsets of K with $\bigcup A_n \cap \bigcup B_n = \emptyset$. Since $K = \mathscr{K}(\mathfrak{A})$, this is equivalent to saying that we work with pairs $((A_n),(B_n))$ with (A_n), (B_n) being disjoint sequences in $\mathfrak{A}$ and $A_m \wedge B_n = 0_{\mathfrak{A}}$ for all m, n.

Specifically, property (K$'$) says that whenever $((A_n),(B_n))$ is such a pair, there exists an infinite $\tau \subseteq \omega$ such that

(i) $\mathfrak{A}$ contains the set $A_\tau = \operatorname{int}\left(\overline{\bigcup_{n\in\tau} A_n}\right)$, which is the supremum of $(A_n)_{n\in\tau}$ in $\mathfrak{R}_{\{0,1\}^{2^\omega}}$ and hence in $\mathfrak{A}$ (here the closure is taken in $\{0,1\}^{2^\omega}$),

(ii) the pair $((B_n)_{n\in\tau},(B_n)_{n\notin\tau})$ forms a *forbidden splitting* in $\mathfrak{A}$, that is, there does not exist $A \in \mathfrak{A}$ with the property that $B_n \leqslant A$ if $n \in \tau$ and $B_n \wedge A = 0_{\mathfrak{A}}$ if $n \notin \tau$.

In the spirit of [Kos04] and [Hay81], the algebra $\mathfrak{A}$ is obtained as the union of an "increasing" transfinite sequence $(\mathfrak{A}_\alpha)_{\alpha<2^\omega}$ of Boolean algebras. Each $\mathfrak{A}_\alpha$ has the following properties:

(a) $\mathfrak{A}_\alpha$ lies between the algebra $\mathfrak{D}_{\{0,1\}^\alpha}$ of clopen subsets of $\{0,1\}^\alpha$ and the algebra $\mathfrak{R}_{\{0,1\}^\alpha}$ of regular open subsets of $\{0,1\}^\alpha$,

(b) the conditions of property (K$'$) are satisfied for a specific pair $((A_n),(B_n)) \subseteq \mathfrak{A}_\alpha \times \mathfrak{A}_\alpha$,

(c) if at stage $\beta < \alpha$ the conditions of property (K$'$) are satisfied for a pair $((A'_n),(B'_n))$, they must still be satisfied at stage α.

The condition (c) may be rephrased as follows. If for some $\beta < \alpha$ we have a sequence $(A_n)_{n\in\tau} \subseteq \mathfrak{A}_\beta$ with $A_\tau \in \mathfrak{A}_\beta$ and a forbidden splitting $(\mathcal{P},\mathcal{Q}) \subseteq \mathfrak{A}_\beta \times \mathfrak{A}_\beta$, then both A_τ and $(\mathcal{P},\mathcal{Q})$ are *preserved* in $\mathfrak{A}_\alpha$.

Condition (b) tells us that in addition to all the suprema and forbidden splittings which had to preserved before and at stage α, we need to add another supremum and forbidden splitting which will have to preserved at and after stage α.

Now, once the construction of $(\mathfrak{A}_\alpha)_{\alpha<2^\omega}$ and $\mathfrak{A}$ has been completed, due to König's Lemma [Kun80, p. 34, Lemma 10.40] it turns out that if $((A_n),(B_n))$ is a pair of disjoint sequences in $\mathfrak{A}$ with $A_m \wedge B_n = 0_{\mathfrak{A}}$ for each m, n, then it must have been dealt with at some stage α. Consequently, there exists an infinite $\tau \subseteq \omega$ such that $A_\tau \in \mathfrak{A}_\alpha$ and $((B_n)_{n\in\tau},(B_n)_{n\notin\tau})$ is forbidden in $\mathfrak{A}_\alpha$. By (c), this means that both A_τ and $((B_n)_{n\in\tau},(B_n)_{n\notin\tau})$ are preserved in $\mathfrak{A}$, and so $\mathfrak{A}$ has property (K$'$). Using a result similar to Rosenthal's lemma, we can show that property (K$'$) prevents $\mathscr{L}^{\mathscr{C}(\mathscr{K}(\mathfrak{A}))}$ from containing noncentripetal operators.

Finally, to conquer (P3), we modify the above construction slightly and instead of adding one pair to the list of forbidden splittings, at each stage we add up to four new pairs. More precisely, we consider a pair $((C_n),(D_n))$ of disjoint sequences in $\mathfrak{A}$ and if it forms a forbidden splitting $(\mathcal{P},\mathcal{Q})$ at stage α, we ensure that $(\mathcal{P},\mathcal{Q})$, $(\mathcal{P}\times\{0\},\mathcal{Q}\times\{0\})$ and $(\mathcal{P}\times\{1\},\mathcal{Q}\times\{1\})$ are all forbidden at stage $\alpha+1$ (and all the subsequent stages).

Again, once the construction has been completed, it can be shown that the condition $\overline{V}\cap\overline{W}\neq\emptyset$ is equivalent to saying that there exist $\alpha<2^\omega$ and a pair $(\mathcal{P},\mathcal{Q})$ which cannot be split in $\mathfrak{A}_\alpha$. However, we ensured that both $(\mathcal{P}\times\{0\},\mathcal{Q}\times\{0\})$ and $(\mathcal{P}\times\{1\},\mathcal{Q}\times\{1\})$ are forbidden in $\mathfrak{A}_{\alpha+1}$ and hence, by construction, in $\mathfrak{A}$. This means that for each $i\in\{0,1\}$ the set $\overline{V}\cap\overline{W}$ contains a point x_i corresponding to the "intersection" of $\bigcup(\mathcal{P}\times\{i\})$ and $\bigcup(\mathcal{Q}\times\{i\})$, and it is evident that $x_0\neq x_1$. Thus K cannot contain open butterflies. The idea is similar to the one exploited in [Kos04, Section 6].

3.1.2 Overview of the chapter

The chapter is organised as follows. We start with collating some introductory material on Boolean algebras (section 3.2.1) and Stone spaces (section 3.2.2). We then proceed to section 3.3 in which we define property (K$'$) and show that the Stone space of an algebra with property (K$'$) is weakly Koszmider. Section 3.4 contains all intermediate results regarding preservation of suprema and forbidden splittings. Finally, section 3.5 is devoted to the construction of an algebra $\mathfrak{A}$ the Stone space of which satisfies properties (P1)–(P3), as is checked in section 3.6.

3.1.3 Notation and terminology

- For any ordinals α, β with $\beta \leqslant \alpha \leqslant 2^\omega$ we define $p^\alpha_\beta : \{0,1\}^\alpha \to \{0,1\}$ to be the natural projection onto the β^{th} coordinate, that is, if $(x_\gamma)_{\gamma<\alpha} \in \{0,1\}^\alpha$, we define

$$p^\alpha_\beta((x_\gamma)_{\gamma<\alpha}) = x_\beta.$$

- For any α, β with $\beta \leqslant \alpha \leqslant 2^\omega$ we define $\pi^\alpha_\beta : \{0,1\}^\alpha \to \{0,1\}^\beta$ to be the natural projection from $\{0,1\}^\alpha$ onto $\{0,1\}^\beta$, that is, if $(x_\gamma)_{\gamma<\alpha} \in \{0,1\}^\alpha$, we define

$$\pi^\alpha_\beta((x_\gamma)_{\gamma<\alpha}) = (x_\gamma)_{\gamma<\beta}.$$

- Consider any $\beta \leqslant \alpha \leqslant 2^\omega$ and suppose that $A \subseteq \{0,1\}^\beta$. We define the *lifting* of A to $\{0,1\}^\alpha$ to be the set

$$A^{[\alpha]} = \left(\pi^\alpha_\beta\right)^{-1}(A).$$

  Similarly, if $\mathcal{A} \subseteq \mathscr{P}(\{0,1\}^\beta)$, we define

$$\mathcal{A}^{[\alpha]} = \{A^{[\alpha]} : A \in \mathcal{A}\}.$$

- Let $\alpha \leqslant 2^\omega$, $\mathcal{A} \subseteq \mathscr{P}(\{0,1\}^\alpha)$ and $I \subseteq \{0,1\}$. We define

$$\mathcal{A} \times I = \{A \times I : A \in \mathcal{A}\}.$$

3.2 Preliminary results

3.2.1 Introduction to Boolean algebras

The purpose of this section is to put together standard results on Boolean algebras which will be used throughout the chapter. For more information on Boolean algebras we refer the reader to [Hal63] or [Sem71].

We start with some standard notation. Recall that a subset A of a topological space K is said to be *regular open* if $\operatorname{int}\left(\overline{A}\right) = A$.

Notation 3.2.1. Let K be a topological space. We write

- $\mathfrak{D}_K$ to denote the algebra of clopen subsets of K with $0_{\mathfrak{D}_K} = \emptyset$, $1_{\mathfrak{D}_K} = K$ and usual set-theoretic operations $A \wedge B = A \cap B$, $A \vee B = A \cup B$.

- $\mathfrak{R}_K$ to denote the algebra of regular open subsets of K with $0_{\mathfrak{R}_K} = \emptyset$, $1_{\mathfrak{R}_K} = K$ and operations defined by $A \wedge B = A \cap B$, $A \vee B = \mathrm{int}\left(\overline{A \cup B}\right)$.

Notation 3.2.2. Let $\mathfrak{A}$ be a Boolean algebra. We write

- $\mathscr{K}(\mathfrak{A})$ to denote the Stone space of $\mathfrak{A}$.

Suppose that $\mathfrak{B}$ is a Boolean algebra with $\mathfrak{A} \subseteq \mathfrak{B}$, and $X \in \mathfrak{B} \backslash \mathfrak{A}$. We write

- $\mathfrak{A} \leqslant \mathfrak{B}$ if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.

- $\langle \mathfrak{A}, X \rangle$ to denote the algebra generated by $\mathfrak{A}$ and X.

It is clear that $\langle \mathfrak{A}, X \rangle$ consists of all finite combinations of X and elements of $\mathfrak{A}$. However, for the purpose of our construction we need a stronger result.

Lemma 3.2.3. *Let $\mathfrak{A}$ be a Boolean algebra. Then*

$$\langle \mathfrak{A}, X \rangle = \{A \vee (A' \wedge X) \vee (A'' \wedge \neg X) : A, A', A'' \textit{ are pairwise disjoint elements of } \mathfrak{A}\}.$$

For the proof of the lemma note that the set written on the right hand side of the above expression is an algebra and must be contained in any algebra containing $\mathfrak{A}$ and X.

We now need to introduce more terminology.

Definition 3.2.4. Let $\mathfrak{A}$ be a Boolean algebra, $A, B \in \mathfrak{A}$, $(A_n) \subseteq \mathfrak{A}$, $\mathcal{P}, \mathcal{Q} \subseteq \mathfrak{A}$ and $\tau \subseteq \omega$. We say that

- $A \leqslant B$ if $A \wedge B = A$ or, equivalently, if $A \vee B = B$,

- (A_n) is *disjoint* if $A_m \wedge A_n = 0_{\mathfrak{A}}$ for all $m \neq n$,

- an element $A \in \mathfrak{A}$ is a *supremum of* $(A_n)_{n \in \tau}$ *in* $\mathfrak{A}$ if A is the supremum of the set $\{A_n : n \in \tau\}$ ordered by $\leqslant$. When a supremum of $(A_n)_{n\in\tau}$ exists in $\mathfrak{A}$, we denote it by $\vee_{n\in\tau} A_n$,

- the pair $(\mathcal{P}, \mathcal{Q})$ is *split in* $\mathfrak{A}$ *(by A)* if there exists $A \in \mathfrak{A}$ such that for each $X \in \mathcal{P} \cup \mathcal{Q}$ we have
$$X \wedge A = \begin{cases} X & \text{if} \quad X \in \mathcal{P} \\ 0_{\mathfrak{A}} & \text{if} \quad X \in \mathcal{Q}, \end{cases}$$

- the pair $(\mathcal{P}, \mathcal{Q})$ *forms a forbidden splitting in* $\mathfrak{A}$ if it cannot be split in $\mathfrak{A}$.

The next piece of notation is nonstandard and hence deserves a separate number.

Notation 3.2.5. Let K be a topological space and suppose that $(A_n)_{n\in\omega} \subseteq K$ is a disjoint sequence. For any $\tau \subseteq \omega$ we define

$$A_\tau = \operatorname{int}\left(\overline{\bigcup_{n\in\tau} A_n}\right),$$

where the closure is taken in K. To avoid ambiguity, we may also use the notation $(A)_\tau$.

Lemma 3.2.6 ([Hal63, Chapter 7, Lemma 1])**.** *Let K be a topological space, $\mathfrak{A} \leqslant \mathfrak{R}_K$ and consider a disjoint sequence $(A_n)_{n\in\omega} \subseteq \mathfrak{A}$ and $\tau \subseteq \omega$. Suppose that $A_\tau \in \mathfrak{A}$. Then $(A_n)_{n\in\tau}$ has a supremum in $\mathfrak{A}$ and*

$$\vee_{n\in\tau} A_n = A_\tau.$$

Of course, given a general Boolean algebra, it may not be straightforward to describe its logic operations, let alone prove results involving them. Fortunately, as was proved by Stone, every Boolean algebra has a relatively simple representation.

Theorem 3.2.7 (Stone Representation Theorem, [Hal63, chapter 18])**.**

(i) Let $\mathfrak{A}$ be a Boolean algebra. There exists an algebra isomorphism between $\mathfrak{A}$ and $\mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$.

(ii) Let K be a zero-dimensional topological space. There exists a homeomorphism between K and $\mathscr{K}(\mathfrak{D}_K)$.

The first clause of this theorem implies, in particular, that in order to prove a result for a general algebra $\mathfrak{A}$, it is sufficient to prove the analogous result for $\mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$, and the latter is usually easier to do, due to simplicity of the structure of $\mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$. We will often appeal to this argument and so, to avoid repetition, let us introduce more notation.

Notation 3.2.8. Let $\mathfrak{A}$ be a Boolean algebra, $A \in \mathfrak{A}$ and $\mathcal{A} \subseteq \mathfrak{A}$. We will write

- $\tilde{}$ for the algebra isomorphism between $\mathfrak{A}$ and $\mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$ arising from the Stone Representation Theorem; to avoid ambiguity, we may also denote this isomorphism by $\tilde{}^{\mathfrak{A}}$,
- $\widetilde{A}$ for the image of A under $\tilde{}$,
- $\widetilde{\mathcal{A}}$ for the set $\{\widetilde{A} : A \in \mathcal{A}\}$.

Let us now mention a few relationships between elements of $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$ which follow immediately from the Stone Representation Theorem or Lemma 3.2.3.

Proposition 3.2.9. *Let $\mathfrak{A}$ be a Boolean algebra, $A, B \in \mathfrak{A}$, $(A_n) \subseteq \mathfrak{A}$ and $\mathcal{P}, \mathcal{Q} \subseteq A$. Then*

(i) $\widetilde{A \vee B} = \widetilde{A} \cup \widetilde{B}$, $\widetilde{A \wedge B} = \widetilde{A} \cap \widetilde{B}$, $\widetilde{0_{\mathfrak{A}}} = \emptyset$ *and* $\widetilde{1_{\mathfrak{A}}} = \mathscr{K}(\mathfrak{A})$*,*

(ii) $A \leqslant B$ *if and only if* $\widetilde{A} \subseteq \widetilde{B}$*,*

(iii) (A_n) *is disjoint if and only if* $(\widetilde{A_n})$ *is disjoint,*

(iv) (A_n) *has a supremum in* $\mathfrak{A}$ *if and only if* $(\widetilde{A_n})$ *has a supremum in* $\widetilde{\mathfrak{A}}$ *and when this is the case,*

$$\vee_{n\in\omega}\widetilde{A_n} = \widetilde{\vee_{n\in\omega}A_n},$$

(v) the pair $(\mathcal{P}, \mathcal{Q})$ *is split in* $\mathfrak{A}$ *by* A *if and only if the pair* $\left(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}}\right)$ *is split in* $\widetilde{\mathfrak{A}}$ *by* $\widetilde{A}$.

Finally, suppose that $\mathfrak{B}$ *is an algebra with* $\mathfrak{A} \leqslant \mathfrak{B}$ *and let* $X \in \mathfrak{B}\backslash\mathfrak{A}$*. Then*

(vi) $\widetilde{\langle \mathfrak{A}, X\rangle}^{\mathfrak{B}} = \left\langle \widetilde{\mathfrak{A}}^{\mathfrak{B}}, \widetilde{X}^{\mathfrak{B}} \right\rangle$.

We finish the section with another well-known result.

Lemma 3.2.10. *Let* K *be a zero-dimensional topological space and consider the algebra* $\mathfrak{A} = \mathfrak{D}_K$*, a disjoint sequence* $(A_n) \subseteq \mathfrak{A}$ *and a pair* $(\mathcal{P}, \mathcal{Q}) \subseteq \mathfrak{A} \times \mathfrak{A}$*. Then*

(i) (A_n) *has a supremum in* $\mathfrak{A}$ *if and only if* $A_\omega \in \mathfrak{A}$*, that is, if and only if* A_ω *is clopen in* K*, and in this case*

$$\vee_{n\in\omega}A_n = A_\omega = \overline{\bigcup_{n\in\omega} A_n},$$

(ii) $(\mathcal{P}, \mathcal{Q})$ *is split in* $\mathfrak{A}$ *if and only if*

$$\overline{\bigcup \mathcal{P}} \cap \overline{\bigcup \mathcal{Q}} = \emptyset.$$

The proof of the lemma is immediate as K has a basis consisting of clopen sets.

3.2.2 Introduction to irreducible maps and Gleason spaces

This section only contains basic facts. For more information we refer the reader to [CN74, Chapter 2].

Definition 3.2.11. Let $f\colon X \twoheadrightarrow Y$ be a continuous surjection. We say that f is *irreducible* if there does not exist a proper closed subset F of X with $f(F) = Y$.

Theorem 3.2.12 (Gleason, [Gle58])**.** *Let K be a topological space. Then $\mathscr{K}(\mathfrak{R}_K)$ is extremally disconnected and there exists a continuous irreducible surjection $f\colon \mathscr{K}(\mathfrak{R}_K) \twoheadrightarrow K$.*

Moreover, if G is another extremally disconnected space with the property that there exists a continuous irreducible surjection $g\colon G \twoheadrightarrow K$, then there exists a homeomorphism $\phi\colon G \to \mathscr{K}(\mathfrak{R}_K)$ with $f\phi = g$.

Definition 3.2.13. Let K be a topological space. Following the notation of Theorem 3.2.12,

- we say that $\mathscr{K}(\mathfrak{R}_K)$ is the *Gleason space of* K and we denote it by $\mathscr{G}_K$,
- we say that f is the *canonical projection* and we denote it by γ_K.

Gleason spaces turn out to be very useful in our context as they preserve separability.

Theorem 3.2.14. *Let X and Y be topological spaces such that there exists a continuous irreducible surjection $\rho\colon X \twoheadrightarrow Y$. Suppose, in addition, that Y is separable. Then X is also separable. In particular, the Gleason space of a separable space is separable.*

Proof. Let $\{y_n : n \in \omega\}$ be a dense subset of Y. For each n pick some $x_n \in \rho^{-1}(\{y_n\})$ and consider the set $D = \{x_n : n \in \omega\}$. If $\overline{D} \neq X$ then, by irreducibility, $Y \backslash \rho(\overline{D})$ is a nonempty open subset of Y and hence contains some y_n. This, however, leads to a contradiction as $y_n = \rho(x_n) \in \rho(D)$. □

We are interested in a specific case of Theorem 3.2.14.

Corollary 3.2.15. *$\mathscr{G}_{\{0,1\}^{2^\omega}}$ is separable.*

Proof. It is known (see, e.g. [Dug66, p. 175, Theorem 7.2]) that the space $\{0,1\}^{2^\omega}$ is separable, and so the result follows immediately from Theorem 3.2.14. □

With one extra step added, Corollary 3.2.15 provides a recipe for obtaining (P1).

Corollary 3.2.16. *Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{R}_{\{0,1\}^{2^\omega}}$. Then the Stone space of $\mathfrak{A}$ is separable.*

Proof. By Stone duality [Hal63, Chapter 20, Theorem 8], saying that $\mathfrak{A} \leqslant \mathfrak{R}_{\{0,1\}^{2^\omega}}$ is equivalent to saying that there exists a continuous surjection $\sigma \colon \mathscr{K}(\mathfrak{R}_{\{0,1\}^{2^\omega}}) \twoheadrightarrow \mathscr{K}(\mathfrak{A})$. Now, $\mathscr{K}(\mathfrak{R}_{\{0,1\}^{2^\omega}}) = \mathscr{G}_{\{0,1\}^{2^\omega}}$ and so is separable. Finally, $\mathscr{K}(\mathfrak{A})$ is a continuous image of a separable space hence, by [Wil70, Theorem 16.4a], is itself separable. □

3.3 Property (K′)

Suppose that we want to check whether a given space K is weakly Koszmider. As a possible solution, we can verify centripetality of all operators on $\mathscr{C}(K)$ or check commutativity of $\mathscr{L}/\mathscr{L}_{\mathrm{wc}}$. In their papers [Kos04] and [Ple04], Koszmider and Plebanek take a different approach. Plebanek, in particular, introduced a property (H) which relies on topological properties of K and showed that any space with this property is weakly Koszmider. As mentioned in the introduction, the adaptation of this property to zero-dimensional spaces (called property (H′)) is, in turn, a modification of the Subsequential Completeness property introduced by Haydon in [Hay81].

We are now going to present a modification of property (H′). For the purpose of this section we restrict ourselves to a specific class of Boolean algebras.

Definition 3.3.1. Let K be a topological space and suppose that $\mathfrak{A} \leqslant \mathfrak{R}_K$. We say that $\mathfrak{A}$ *has property (K′)* if, given

(a) a disjoint sequence (A_n) of nonempty elements of $\mathfrak{A}$, and

(b) a disjoint sequence (B_n) of nonempty elements of $\mathfrak{A}$ with $A_m \wedge B_n = 0_{\mathfrak{A}}$ for all m, n,

there exists an infinite $\tau \subseteq \omega$ such that

(i) $A_\tau \in \mathfrak{A}$ (and so $(A_n)_{n\in\tau}$ has a supremum in $\mathfrak{A}$), and

(ii) the pair $((B_n)_{n\in\tau}, (B_n)_{n\notin\tau})$ cannot be split in $\mathfrak{A}$.

Theorem 3.3.2. *Let K be a topological space and suppose that a subalgebra $\mathfrak{A}$ of $\mathfrak{R}_K$ has property (K′). Then the Stone space $\mathscr{K}(\mathfrak{A})$ of $\mathfrak{A}$ is weakly Koszmider.*

Instead of working with a generic algebra $\mathfrak{A}$, we switch to $\mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$. For this reason let us temporarily abuse the existing notation and make an assumption which will allow us to write out arguments in a simpler and more aesthetically pleasing manner.

Assumption 3.3.3 (Valid until the rest of section 3.3)**.** Throughout the rest of this section

(i) unless stated otherwise, all cited subsets of ω (such as τ, σ etc.) are assumed to be infinite,

(ii) unless stated otherwise, all cited subsets of K of the form A_n, B_{n_i}, X, Y and similar (but *not* of the form A_τ) are assumed to be nonempty and clopen,

(iii) if $(A_n) \subseteq K$, we say that $(B_n) \subseteq K$ is a *refinement* of (A_n) if $B_n \subseteq A_n$ for each n. When this is the case, we write $(B_n) \preccurlyeq (A_n)$. As discussed above, it is implicitly assumed that all A_n and B_n are nonempty and clopen.

The proof of Theorem 3.3.2 goes along the lines of the corresponding proofs in [Kos04] and [Ple04]. The principal ingredient in the cited papers is Rosenthal's lemma [Die84] and we need something similar.

Lemma 3.3.4. *Let K be a zero-dimensional space and suppose that we are given*

(a) an operator $T\colon \mathscr{C}(K) \to \mathscr{C}(K)$,

(b) a sequence (A_n) of mutually disjoint (nonempty clopen) subsets of K,

(c) a sequence (B_n) of mutually disjoint subsets of K,

(d) $\varepsilon > 0$.

Then there exist

(i) (an infinite) $\sigma \subseteq \omega$,

(ii) a refinement $(D_n)_{n\in\sigma} \preccurlyeq (B_n)_{n\in\sigma}$

such that whenever τ is a subset of σ with A_τ clopen and $n \in \sigma \backslash \tau$, we have

$$\left\| [T\chi_{A_\tau}]|_{D_n} \right\|_\infty \leqslant \varepsilon.$$

The proof of Lemma 3.3.4, in turn, requires a few more intermediate results which we write out as separate claims.

Claim 3.3.5. *Let $g \in \mathscr{C}(K)$ and suppose that there exist $X \subseteq K$ and $\delta > 0$ with*

$$\| [g]|_X \|_\infty > \delta.$$

Then we can find $Y \subseteq X$ and $\alpha \in \{-1, 1\}$ such that

$$[\alpha g]|_Y > \delta.$$

Proof. Let $x \in X$ be such that $|g(x)| > \delta$. Taking $\alpha = \operatorname{sign}(g(x))$ we get $g(\alpha x) > \delta$ and so $x \in (\alpha g)^{-1}((\delta, \infty))$. But $(\alpha g)^{-1}((\delta, \infty))$ is open and hence there exists $Y \subseteq X$ with $x \in Y \subseteq (\alpha g)^{-1}((\delta, \infty))$ as required. □

Claim 3.3.6. *Let $(X_n) \subseteq K$ be a disjoint sequence and consider any $\tau, \theta \subseteq \omega$ with X_τ and X_θ clopen. Then*

(i) $X_{\tau \cap \theta} = X_\tau \cap X_\theta$.

Suppose now that $F \subseteq \tau$ is finite. Then

(ii) X_F is clopen in K and $X_F = \bigcup_{n \in F} X_n$,

(iii) X_τ is clopen in K and $X_\tau = X_{\tau \setminus F} \cup X_F$, where the union is disjoint.

Proof. (i) First of all note that $X_\tau \cap X_\theta$ is a clopen set containing $\bigcup_{n \in \tau \cap \theta} X_n$ and so

$$X_\tau \cap X_\theta \supseteq \operatorname{int}\left(\overline{\bigcup_{n \in \tau \cap \theta} X_n} \right) = X_{\tau \cap \theta}.$$

On the other hand, however,

$$\begin{aligned}
X_\tau \cap X_\theta &= \operatorname{int}\left(\overline{\bigcup_{n \in \tau} X_n} \right) \cap \operatorname{int}\left(\overline{\bigcup_{n \in \theta} X_n} \right) \\
&\subseteq \operatorname{int}\left(\overline{\bigcup_{n \in \tau} X_n \cap \bigcup_{n \in \theta} X_n} \right) \qquad \text{(mentioned in [Ple04])} \\
&= \operatorname{int}\left(\overline{\bigcup_{n \in \tau \cap \theta} X_n} \right) \\
&= X_{\tau \cap \theta}.
\end{aligned}$$

(ii) The proof is immediate as $\bigcup_{n\in F} X_n$ is clopen.

(iii) By above and Lemma 3.2.10, it is sufficient to show that

$$\overline{\bigcup_{n\in\tau} X_n}\setminus\bigcup_{n\in F} X_n = \overline{\bigcup_{n\in\tau\setminus F} X_n}.$$

Let $x \in \overline{\bigcup_{n\in\tau} X_n}\setminus\bigcup_{n\in F} X_n$ and $U \ni x$ be open. Then $U\setminus\bigcup_{n\in F} X_n$ is an open set containing x and thus $(U\setminus\bigcup_{n\in F} X_n)\cap X_n \neq \emptyset$ for some $n \in \tau$. Clearly, $n \notin F$ which implies that $x \in \overline{\bigcup_{n\in\tau\setminus F} X_n}$. The reverse inclusion follows from disjointness of (X_n). □

Claim 3.3.7. *Suppose that we are given an operator $S\colon \mathscr{C}(K) \to \mathscr{C}(K)$, disjoint sequences $(X_n), (Y_n) \subseteq \mathscr{P}(K)$ and $\delta > 0$. Then*

(i) there exist $M \in \omega$ and a refinement $(Y_n^{(1)}) \preccurlyeq (Y_n)$ such that

$$\left\| [S\chi_{X_M}]|_{Y_n^{(1)}} \right\|_\infty \leqslant \delta \quad \textit{for infinitely many } n,$$

(ii) for any $N \in \omega$ there exist $\sigma_N \subseteq \omega$ and $Y_N^{(2)} \subseteq Y_N$ such that whenever τ is a subset of σ_N with X_τ clopen, we have

$$\left\| [S\chi_{X_\tau}]|_{Y_N^{(2)}} \right\|_\infty \leqslant \delta.$$

Proof. (i) Suppose, for a contradiction, that the statement is false. Then for any $M \in \omega$ and $(Y_n^{(1)}) \preccurlyeq (Y_n)$ there exists k such that

$$\left\| [S\chi_{X_M}]|_{Y_n^{(1)}} \right\|_\infty > \delta \quad \forall n \geqslant k.$$

Consequently, by Claim 3.3.5, for each $n \geqslant k$ there exist $\alpha \in \{-1, 1\}$ and $\hat{Y}_n^{(1)} \subseteq Y_n^{(1)}$ with

$$[S(\alpha\chi_{X_M})]|_{\hat{Y}_n^{(1)}} > \delta. \tag{3.1}$$

To get a contradiction, we construct inductively sequences $(k_j)_{j\in\omega}$, $((\alpha_{j,n})_{n\geqslant k_j})_{j\in\omega}$ and $((Y_{j,n})_{n\geqslant k_j})_{j\in\omega}$ with the property that for each j

(i.a) $k_{j+1} > k_j$,

(i.b) $(\alpha_{j,n})_{n\geqslant k_j} \subseteq \{-1, 1\}$,

(i.c) $(Y_{j+1,n})_{n\geqslant k_{j+1}} \preccurlyeq (Y_{j,n})_{n\geqslant k_{j+1}} \preccurlyeq (Y_n)_{n\geqslant k_{j+1}}$, (i.d) $\left[S(\alpha_{j,n}\chi_{X_j})\right]\big|_{Y_{j,n}} > \delta$ for all $n \geqslant k_j$.

Base Case. By (3.1), there exists $k_0 \in \omega$ such that for each $n \geqslant k_0$ we can find $\alpha_{0,n} \in \{-1, 1\}$ and $Y_{0,n} \subseteq Y_n$ such that

$$[S\left(\alpha_{0,n}\chi_{X_0}\right)]|_{Y_{0,n}} > \delta.$$

Inductive Step. Suppose that the construction has been completed up to some j. Appealing to (3.1), we can pick $k \in \omega$ such that for each $n \geqslant k$ there exist $\alpha_{j+1,n} \in \{-1, 1\}$ and $Y_{j+1,n} \subseteq Y_{j,n}$ such that

$$\left[S\left(\alpha_{j+1,n}\chi_{X_{j+1}}\right)\right]\big|_{Y_{j+1,n}} > \delta.$$

To complete the inductive step, take $k_{j+1} = \max\{k_j + 1, k\}$ to ensure $k_{j+1} > k_j$.

Once the construction has been completed, take $J = \left\lceil \frac{\|S\|}{\delta} \right\rceil$. It follows from the disjointness of (X_n) that any function of the form $\pm\chi_{X_0} \pm \cdots \pm \chi_{X_J}$ has norm 1 and, in particular,

$$\|S(\alpha_{0,k_J}\chi_{X_0} + \cdots + \alpha_{J,k_J}\chi_{X_J})\|_\infty \leqslant \|S\|.$$

On the other hand, since $(k_j)_{0\leqslant j\leqslant J}$ is an increasing sequence, while $(Y_{j,k_J})_{0\leqslant j\leqslant J}$ is decreasing, condition (i.d) implies that

$$[S(\alpha_{0,k_J}\chi_{X_0} + \cdots + \alpha_{J,k_J}\chi_{X_J})]|_{Y_{J,k_J}} > (J+1)\delta > \frac{\|S\|}{\delta}\delta = \|S\|,$$

which, combined with the previous estimate, gives a contradiction.

(ii) Fix $N \in \omega$ and suppose, for a contradiction, that for any $\sigma \subseteq \omega$ and $Y_N^{(2)} \subseteq Y_N$ there exists $\tau \subseteq \sigma$ such that X_τ is clopen and

$$\left\| [S\chi_{X_\tau}]|_{Y_N^{(2)}} \right\|_\infty > \delta.$$

Consequently, we can find $\alpha \in \{-1, 1\}$ and $\hat{Y}_N^{(2)} \subseteq Y_N^{(2)}$ such that

$$[S\left(\alpha\chi_{X_\tau}\right)]|_{\hat{Y}_N^{(2)}} > \delta. \tag{3.2}$$

Define $J = \left\lceil \frac{\|S\|}{\delta} \right\rceil$ and partition ω into $J + 1$ disjoint infinite parts $\sigma_0, \ldots, \sigma_J$. Using another inductive argument, we construct sequences $(\tau_j)_{0\leqslant j\leqslant J}$, $(\alpha_j)_{0\leqslant j\leqslant J}$ and $(Y_{j,N})_{0\leqslant j\leqslant J}$ such that for each j

(ii.a) $\tau_j \subseteq \sigma_j$,

(ii.b) $\alpha_j \in \{-1, 1\}$,

(ii.c) $Y_{j+1,N} \subseteq Y_{j,N} \subseteq Y_N$,

(ii.d) X_{τ_j} is clopen and $\left[S\left(\alpha_j \chi_{X_{\tau_j}}\right)\right]\Big|_{Y_{j,N}} > \delta$.

Base Case follows from (3.2) applied to σ_0 and Y_N. For the *Inductive Step* suppose that the construction has been completed up to stage j with $0 \leqslant j < J$ and apply (3.2) to σ_{j+1} and $Y_{j,N}$.

We now proceed as in part (i). Disjointness of $(\sigma_j)_{0\leqslant j\leqslant J}$ and Claim 3.3.6 imply that $(X_{\tau_j})_{0\leqslant j\leqslant J}$ is disjoint and so

$$\left\|S\left(\alpha_0\chi_{X_{\tau_0}} + \cdots + \alpha_J\chi_{X_{\tau_J}}\right)\right\|_\infty \leqslant \|S\| \left\|\alpha_0\chi_{X_{\tau_0}} + \cdots + \alpha_J\chi_{X_{\tau_J}}\right\|_\infty = \|S\|,$$

while, on the other hand, since $(Y_{j,N})_{0\leqslant j\leqslant J}$ is a decreasing sequence, condition (ii.d) implies that

$$\left[S\left(\alpha_0\chi_{X_{\tau_0}} + \cdots + \alpha_J\chi_{X_{\tau_J}}\right)\right]\Big|_{Y_{J,N}} > (J+1)\delta > \frac{\|S\|}{\delta}\delta = \|S\|$$

giving the required contradiction. □

Proof of Lemma 3.3.4. The proof is constructive and we start with showing that there exist a subsequence (n_j) and a refinement $(C_{n_j}) \preccurlyeq (B_{n_j})$ such that

$$\left\|\left[T\chi_{A_{n_k}}\right]\Big|_{C_{n_l}}\right\|_\infty \leqslant \frac{\varepsilon}{2^{k+2}} \quad \forall k,l : k < l. \tag{3.3}$$

To do this, we construct inductively sequences $(n_j)_{j\in\omega}$, $(\Lambda_j)_{j\in\omega}$ and $((B_{j,n})_{n\in\Lambda_j})_{j\in\omega}$ such that for each j

(i.a) $n_{j+1} > n_j$,

(i.b) $\Lambda_{j+1} \subseteq \Lambda_j \subseteq \omega$,

(i.c) $\Lambda_j \ni n_{j+1}$,

(i.d) $n_j < \min \Lambda_j$,

(i.e) $(B_{j+1,n})_{n\in\Lambda_{j+1}} \preccurlyeq (B_{j,n})_{n\in\Lambda_{j+1}} \preccurlyeq (B_n)_{n\in\Lambda_{j+1}}$, (i.f) $\left\| \left[T\chi_{A_{n_j}}\right]\Big|_{B_{j,n}} \right\|_\infty \leqslant \dfrac{\varepsilon}{2^{j+2}}$ for all $n \in \Lambda_j$.

Base Case. By Claim 3.3.7 (i), there exist $n_0 \in \omega$ and $(B_{0,n}) \preccurlyeq (B_n)$ such that

$$\left\| \left[T\chi_{A_{n_0}}\right]\big|_{B_{0,n}} \right\|_\infty \leqslant \frac{\varepsilon}{4} \quad \text{for infinitely many } n. \tag{3.4}$$

We also define

$$\Lambda_0 = \{n > n_0 : (3.4) \text{ holds for } n\}.$$

Inductive Step. Suppose that the construction has been carried out up to some j. Applying part (i) of Claim 3.3.7 to $(A_n)_{n\in\Lambda_j}$ and $(B_{j,n})_{n\in\Lambda_j}$, we can find $n_{j+1} \in \Lambda_j$ (in particular, $n_{j+1} > n_j$) and a refinement $(B_{j+1,n})_{n\in\Lambda_j} \preccurlyeq (B_{j,n})_{n\in\Lambda_j}$ such that

$$\left\| \left[T\chi_{A_{n_{j+1}}}\right]\Big|_{B_{j+1,n}} \right\|_\infty \leqslant \frac{\varepsilon}{2^{j+3}} \quad \text{for infinitely many } n \in \Lambda_j. \tag{3.5}$$

To complete the inductive step, we define

$$\Lambda_{j+1} = \{n \in \Lambda_j : n > n_{j+1} \text{ and } (3.5) \text{ holds for } n\}.$$

Once the construction has been completed, for each j we define

$$C_{n_j} = B_{j,n_j}.$$

Note that if we pick any k, l with $k < l$, then $n_l \in \Lambda_{l-1} \subseteq \Lambda_k$ and so

$$\left\| \left[T\chi_{A_{n_k}}\right]\Big|_{C_{n_l}} \right\|_\infty = \left\| \left[T\chi_{A_{n_k}}\right]\Big|_{B_{l,n_l}} \right\|_\infty \leqslant \left\| \left[T\chi_{A_{n_k}}\right]\Big|_{B_{k,n_l}} \right\|_\infty \leqslant \frac{\varepsilon}{2^{k+2}}$$

giving (3.3).

We now need to further reduce the size of (C_{n_j}) and for this we construct inductively a subsequence (n_{j_r}) of (n_j) and sequences (σ_r) and $(D_{n_{j_r}})$ with the property that for each r

(ii.a) $n_{j_{r+1}} > n_{j_r}$,

(ii.b) $\sigma_{r+1} \subseteq \sigma_r \subseteq (n_j)$,

(ii.c) $n_{j_{r+1}} \in \sigma_r$,

(ii.d) $D_{n_{j_r}} \subseteq C_{n_{j_r}}$, (ii.e) whenever $\tau \subseteq \sigma_r$ is such that A_τ is clopen, we have

$$\left\| [T\chi_{A_\tau}]|_{D_{n_{j_r}}} \right\|_\infty \leqslant \frac{\varepsilon}{2}. \tag{3.6}$$

For the *Base Case* set $n_{j_0} = n_0$ and apply Claim 3.3.7 (ii) to n_{j_0}, $(A_n)_{n\in(n_j)}$, $(C_n)_{n\in(n_j)}$ and $\varepsilon/2$. For the *Inductive Step* set $n_{j_{r+1}}$ to be the $(r+2)^{\text{nd}}$ element of σ_r (that is, if $\sigma_r = (\sigma_{r,k})_{k\in\omega}$ with $\sigma_{r,k+1} > \sigma_{r,k}$ for each k, then $n_{j_{r+1}} = \sigma_{r,r+1}$) and apply the same result to $n_{j_{r+1}}$, $(A_n)_{n\in\sigma_r}$, $(C_n)_{n\in\sigma_r}$ and $\varepsilon/2$.

Once the construction has been completed, we define

$$\sigma = (n_{j_r})_{r\in\omega}.$$

Suppose that τ is a subset of σ with A_τ clopen and $n \in \sigma\backslash\tau$. There exists $l \in \omega$ with

$$n = n_{j_l}.$$

Define

$$\tau' = \tau \cap (n_{j_r})_{r>l}.$$

Then $\tau' \subseteq \sigma_l$ and $\tau\backslash\tau' \subseteq (n_{j_r})_{r<l}$. By Claim 3.3.6, $A_{\tau'}$ and $A_{\tau\backslash\tau'}$ are clopen and

$$\chi_{A_\tau} = \chi_{A_{\tau'}} + \sum_{m\in\tau\backslash\tau'} \chi_{A_m}.$$

Putting together all of the above, we get

$$\begin{aligned}
\left\| [T\chi_{A_\tau}]|_{D_n} \right\|_\infty &= \left\| \left[T \left(\chi_{A_{\tau'}} + \sum_{m\in\tau\backslash\tau'} \chi_{A_m} \right) \right] \Big|_{D_n} \right\|_\infty \\
&\leqslant \left\| [T\chi_{A_{\tau'}}]|_{D_{n_{j_l}}} \right\|_\infty + \sum_{\substack{n_{j_k}\in\tau\backslash\tau' \\ k<l}} \left\| \left[T\chi_{A_{n_{j_k}}}\right]\Big|_{C_{n_{j_l}}} \right\|_\infty \\
&\leqslant \underbrace{\frac{\varepsilon}{2}}_{\text{by (3.6)}} + \underbrace{\sum_{\substack{n_{j_k}\in\tau\backslash\tau' \\ k<l}} \frac{\varepsilon}{2^{j_k+2}}}_{\text{by (3.3), as } k<l \Leftrightarrow j_k<j_l} \\
&\leqslant \frac{\varepsilon}{2} + \sum_{\substack{n_{j_k}\in\tau\backslash\tau' \\ k<l}} \frac{\varepsilon}{2^{k+2}} \\
&< \frac{\varepsilon}{2} + \sum_{k\in\omega} \frac{\varepsilon}{2^{k+2}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

as required. $\square$

We need one more result before we can proceed to the proof of Theorem 3.3.2.

Lemma 3.3.8. *Let K be a zero-dimensional topological space and suppose that an operator $T\colon \mathscr{C}(K) \to \mathscr{C}(K)$ is noncentripetal. Then*

(i) there exist $(x_n) \subseteq K$, a disjoint $(A_n) \subseteq \mathscr{P}(K)$ and $\varepsilon > 0$ such that

$$x_n \notin A_n \quad \textit{and} \quad |(T\chi_{A_n})(x_n)| > \varepsilon \quad \forall n,$$

(ii) there exist disjoint $(A_n), (B_n) \subseteq \mathscr{P}(K)$ and $\varepsilon > 0$ such that

$$A_m \cap B_n = \emptyset \quad \textit{and} \quad \big|[T\chi_{A_n}]|_{B_n}\big| > \varepsilon \quad \forall m, n.$$

Proof. (i) is just negation of part (iii) of Lemma 2.3.2.

(ii) Let (x_n), (A_n) and ε be as in part (i). By disjointness, the norm of any finite sum of $\pm\chi_{A_n}$ is at most 1 hence, by boundedness of T, for any n there exist only finitely many m with $x_n = x_m$. Thus, without loss of generality, we may assume that all x_n are distinct.

Pick any $n \in \omega$. Since $T\chi_{A_n}$ is continuous, we can find C_n with $x_n \in C_n \subseteq K \backslash A_n$ and

$$\big|[T\chi_{A_n}]|_{C_n}\big| > \varepsilon.$$

If $A_m \cap C_n = \emptyset$ for all m, n, we set $B_n = C_n$ for each n. Else there are two possibilities.

Case 1. There exist $m \in \omega$ and a subsequence (n_j) such that

$$A_m \cap C_{n_j} \neq \emptyset \quad \forall j.$$

In this case for each j we define

$$B_{n_j} = A_m \cap C_{n_j}.$$

Then (A_{n_j}) is a disjoint sequence with

$$\left|\left[T\chi_{A_{n_j}}\right]\Big|_{B_{n_j}}\right| > \varepsilon$$

for each j. Furthermore, since $A_{n_j} \cap C_{n_j} = \emptyset$, we have $m \notin \{n_j : j \in \omega\}$. Consequently,

$$\bigcup A_{n_j} \cap \bigcup B_{n_j} \subseteq \bigcup A_{n_j} \cap A_m = \emptyset,$$

and so (A_{n_j}), (B_{n_j}) provide the required pair.

Case 2. For each m there exists N_m such that

$$A_m \cap C_n = \emptyset \quad \forall n \geqslant N_m.$$

In this case we start with constructing inductively a subsequence (n_j) such that

$$A_{n_k} \cap C_{n_l} = \emptyset \quad \forall j, k, l : k \leqslant l \leqslant j. \tag{3.7}$$

For this we define $n_0 = 0$ and, assuming that $n_0 < n_1 < \cdots < n_j$ have been constructed, set $n_{j+1} = \max\{N_{n_1}, \ldots, N_{n_j}, n_j + 1\}$.

Next, we construct inductively sequences (j_i), (J_i) and $(D_{n_{j_i}})$ such that for each i

(a) $j_i \leqslant J_i < j_{i+1}$,

(b) $D_{n_{j_i}} \subseteq C_{n_{j_i}}$,

(c) $A_{n_j} \cap D_{n_{j_i}} = \emptyset$ for all $j > J_i$.

This would mean that if k, l are any numbers with $k > l$, then $j_k \geqslant j_{l+1} > J_l$ and so

$$A_{n_{j_k}} \cap D_{n_{j_l}} = \emptyset, \tag{3.8}$$

which, combined with (3.7), would give the required result.

Base Case. Define $j_0 = 0$. If $A_{n_j} \cap C_{n_{j_0}} = \emptyset$ for all $j > 0$, set $J_0 = j_0$ and $D_{n_{j_0}} = C_{n_{j_0}}$. Otherwise we set J_0 to be any index with $J_0 > 0$ and $A_{n_{J_0}} \cap C_{n_{j_0}} \neq \emptyset$ and we define $D_{n_{j_0}} = A_{n_{J_0}} \cap C_{n_{j_0}}$. Note that if $j > J_0$ then

$$A_{n_j} \cap D_{n_{j_0}} \subseteq A_{n_j} \cap A_{n_{J_0}} = \emptyset.$$

Inductive Step. Suppose that the construction has been completed up to the stage i and define $j_{i+1} = J_i + 1$. Similar to the above, two cases are possible. If $A_{n_j} \cap C_{n_{j_{i+1}}} = \emptyset$ for all $j > j_{i+1}$, we define $J_{i+1} = j_{i+1}$ and $D_{n_{j_{i+1}}} = C_{n_{j_{i+1}}}$. Otherwise we set J_{i+1} to be any index with $J_{i+1} > j_{i+1}$ and $A_{n_{J_{i+1}}} \cap C_{n_{j_{i+1}}} \neq \emptyset$ and define $D_{n_{j_{i+1}}} = A_{n_{J_{i+1}}} \cap C_{n_{j_{i+1}}}$. Then condition (c) follows from disjointness of (A_n). In either case the conditions (a)–(c) of the inductive hypothesis are satisfied. $\square$

Proof of Theorem 3.3.2. Suppose that an operator $T \in \mathscr{L}^{\mathscr{C}(\mathscr{K}(\mathfrak{A}))}$ is noncentripetal. By Lemma 3.3.8, we can find $\varepsilon > 0$ and disjoint $(A_n), (B_n) \subseteq \mathscr{P}(\mathscr{K}(\mathfrak{A}))$ such that

$$A_m \cap B_n = \emptyset \quad \text{and} \quad \big|[T\chi_{A_n}]|_{B_n}\big| > \varepsilon \quad \forall m, n.$$

By Lemma 3.3.4, there exist $\sigma \subseteq \omega$ and $(D_n)_{n\in\sigma} \preccurlyeq (B_n)_{n\in\sigma}$ such that for any subset τ of σ with A_τ clopen and for any $n \in \sigma\backslash\tau$ we have

$$\big|[T\chi_{A_\tau}]|_{D_n}\big| \leqslant \frac{\varepsilon}{3}. \tag{3.9}$$

Now, since $\mathfrak{A}$ has property (K′), it follows that so does $\mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$. Thus, using Lemma 3.2.10, we can find $\tau \subseteq \sigma$ such that

(i) A_τ is clopen,

(ii) $\overline{\bigcup_{n\in\tau} D_n} \cap \overline{\bigcup_{n\in\sigma\backslash\tau} D_n} \neq \emptyset$.

Note that (3.9) implies that

$$\overline{\bigcup_{n\in\sigma\backslash\tau} D_n} \subseteq (T\chi_{A_\tau})^{-1}([-\varepsilon/3, \varepsilon/3]). \tag{3.10}$$

On the other hand, let $n \in \tau$. If we define $\tau' = \tau\backslash\{n\}$ then $A_{\tau'}$ is clopen and so, by (3.9),

$$\Big|\,[T\chi_{A_{\tau'}}]\big|_{D_n}\Big| \leqslant \frac{\varepsilon}{3}.$$

However, we have

$$\big|[T\chi_{A_n}]|_{D_n}\big| > \varepsilon$$

and so

$$\big|[T\chi_{A_\tau}]|_{D_n}\big| = \Big|\,[T\chi_{A_{\tau'}}]\big|_{D_n} + [T\chi_{A_n}]|_{D_n}\Big| > \varepsilon - \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

implying that

$$\overline{\bigcup_{n\in\tau} D_n} \subseteq (T\chi_{A_\tau})^{-1}([-\infty, -2\varepsilon/3] \cup [2\varepsilon/3, \infty]). \tag{3.11}$$

Combination of (3.10) and (3.11) gives that

$$\overline{\bigcup_{n\in\tau} D_n} \cap \overline{\bigcup_{n\in\sigma\backslash\tau} D_n} = \emptyset$$

leading to a contradiction to the condition (ii). Hence our original assumption was wrong and every operator on $\mathscr{C}(\mathscr{K}(\mathfrak{A}))$ is indeed centripetal. □

3.4 Preservation of suprema and forbidden splittings

Let us start with a definition.

Definition 3.4.1. Let α, β be ordinals with $\beta \leqslant \alpha \leqslant 2^\omega$, and suppose that $\mathfrak{A}_\beta \leqslant \mathfrak{R}_{\{0,1\}^\beta}$ and $\mathfrak{A}_\alpha \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$ are such that $\mathfrak{A}_\beta^{[\alpha]} \leqslant \mathfrak{A}_\alpha$.

- Suppose that $(A_n) \subseteq \mathfrak{A}_\beta$ is a disjoint sequence which has a supremum in $\mathfrak{A}_\beta$. We say that *the supremum of* (A_n) *is preserved in* $\mathfrak{A}_\alpha$ if
  - $(A_n^{[\alpha]})$ has a supremum in $\mathfrak{A}_\alpha$, and
  - $\vee_{n\in\omega}(A_n^{[\alpha]}) = (\vee_{n\in\omega}A_n)^{[\alpha]}$.
- Suppose that $(\mathcal{P}, \mathcal{Q}) \subseteq \mathfrak{A}_\beta \times \mathfrak{A}_\beta$ is a forbidden splitting in $\mathfrak{A}_\beta$. We say that $(\mathcal{P}, \mathcal{Q})$ *is preserved in* $\mathfrak{A}_\alpha$ if $\left(\mathcal{P}^{[\alpha]}, \mathcal{Q}^{[\alpha]}\right)$ is a forbidden splitting in $\mathfrak{A}_\alpha$.

Notation 3.4.2. To avoid too many brackets, if the supremum of $(A_n) \subseteq \mathfrak{A}_\beta$ is preserved in $\mathfrak{A}_\alpha$, we will write $\vee_{n\in\omega}A_n^{[\alpha]}$ to denote both $\vee_{n\in\omega}(A_n^{[\alpha]})$ and $(\vee_{n\in\omega}A_n)^{[\alpha]}$.

In particular, if the supremum of (A_n) in $\mathfrak{A}_\beta$ is A_ω and the supremum of $(A_n^{[\alpha]})$ in $\mathfrak{A}_\alpha$ is $\left(A^{[\alpha]}\right)_\omega$ and $(A_\omega)^{[\alpha]} = \left(A^{[\alpha]}\right)_\omega$, we will write $A_\omega^{[\alpha]}$ to denote both of these values.

As described in the introduction, our separable Koszmider space is constructed by means of transfinite induction and at each stage we need to preserve certain suprema of disjoint sequences and forbidden splittings. The next two propositions show that this idea can be implemented at successor stages while the last result of the section ensures that the algorithm can also be performed at limit stages.

Proposition 3.4.3. *Let* $\alpha < 2^\omega$ *be an infinite ordinal and consider a Boolean algebra* $\mathfrak{A}$ *with the property that*

(a) $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{A} \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$,

(b) $|\mathfrak{A}| = \mu < 2^\omega$.

Suppose also that we are given

(c) a disjoint sequence (A_n) *of nonempty elements of* $\mathfrak{A}$,

(d) a disjoint sequence (B_n) *of nonempty elements of* $\mathfrak{A}$ *with* $A_m \wedge B_n = \emptyset$ *for all* m, n,

(e) a disjoint sequence $(X_n) \subseteq \mathfrak{A}$ *with* $X_\omega \in \mathfrak{A}$*,*

(f) an ordinal $\nu < 2^\omega$ *and a transfinite sequence* $((\mathcal{P}_\beta, \mathcal{Q}_\beta))_{\beta<\nu}$ *of forbidden splittings in* $\mathfrak{A}$*.*

For each $\tau \subseteq \omega$ *we define*

$$\mathfrak{A}(\tau) = \langle \mathfrak{A}, A_\tau \rangle .$$

Then for each $\tau \subseteq \omega$

(i) $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{A}(\tau) \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$*,*

(ii) $|\mathfrak{A}(\tau)| < 2^\omega$*,*

(iii) the supremum X_ω *of* $(X_n)_{n\in\omega}$ *is preserved in* $\mathfrak{A}(\tau)$*,*

(iv) A_τ *is a supremum of* $(A_n)_{n\in\tau}$ *in* $\mathfrak{A}(\tau)$*,*

In addition to the above, there exists $\sigma \subseteq \omega$ *such that for any* $\tau \subseteq \sigma$

(v) the algebra $\mathfrak{A}(\tau)$ *preserves all* $(\mathcal{P}_\beta, \mathcal{Q}_\beta)$*.*

Finally, with σ *as in (v), there exists* $\tau \subseteq \sigma$ *such that*

(vi) the pair $((B_n)_{n\in\tau}, (B_n)_{n\notin\tau})$ *cannot be split in* $\mathfrak{A}(\tau)$*.*

Proof. (i) It is clear that $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{A}(\tau)$. For the second inclusion we use the result from [CN74, Chapter 2, Theorem 2.35 (a)] which says that every subset $\mathcal{A}$ of $\mathfrak{R}_{\{0,1\}^\alpha}$ has a supremum (that is, $\mathfrak{R}_{\{0,1\}^\alpha}$ is a *complete* algebra) and moreover $\vee\mathcal{A} = \operatorname{int}\left(\overline{\bigcup \mathcal{A}}\right)$. Thus, in particular, $A_\tau \in \mathfrak{R}_{\{0,1\}^\alpha}$ and $\mathfrak{A}(\tau) \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$ as required.

(ii) follows immediately as $|\mathfrak{A}(\tau)| = |\mathfrak{A}| < 2^\omega$.

(iii) follows from Lemma 3.2.6 as $X_\omega \in \mathfrak{A} \subseteq \mathfrak{A}(\tau)$.

(iv) also follows from Lemma 3.2.6.

(v) The proof of this part is very similar to the analogous proofs in [Hay81] and [Kos04].

Suppose that the statement is false and let $(\sigma_\xi)_{\xi<2^\omega}$ be a family of almost disjoint infinite subsets of ω (for a proof of existence of such a family see [Kun80, p.48, Theorem 1.3]). Then for each $\xi < 2^\omega$ there exist $\tau_\xi \subseteq \sigma_\xi$ and $\beta < \nu$ such that $(\mathcal{P}_\beta, \mathcal{Q}_\beta)$ is split in $\mathfrak{A}(\tau_\xi)$. Let us now switch to the algebra which is dual to $\mathfrak{R}_{\{0,1\}^\alpha}$. Writing, for convenience, $\sim$

instead of ${}^{\sim}\mathfrak{R}_{\{0,1\}^\alpha}$, the last statement is equivalent to saying that $\left(\widetilde{P_\beta}, \widetilde{Q_\beta}\right)$ is split in $\widetilde{\mathfrak{A}}(\tau_\xi) = \left\langle \widetilde{\mathfrak{A}}, \widetilde{A_{\tau_\xi}} \right\rangle$. This, in turn, means that we can find disjoint $A, A', A'' \in \mathfrak{A}$ such that

$$\bigcup \widetilde{\mathcal{P}_\beta} \subseteq \widetilde{A} \cup \left(\widetilde{A'} \cap \widetilde{A_{\tau_\xi}}\right) \cup \left(\widetilde{A''} \backslash \widetilde{A_{\tau_\xi}}\right), \tag{3.12}$$

$$\bigcup \widetilde{\mathcal{Q}_\beta} \cap \left[\widetilde{A} \cup \left(\widetilde{A'} \cap \widetilde{A_{\tau_\xi}}\right) \cup \left(\widetilde{A''} \backslash \widetilde{A_{\tau_\xi}}\right)\right] = \emptyset. \tag{3.13}$$

However, there are only μ possibilities for picking an element of $\mathfrak{A}$ and $|\nu|$ possibilities for picking β. Consequently, there are at most $\max\{\mu, |\nu|\} < 2^\omega$ choices for picking a quadruplet (A, A', A'', β) versus 2^ω choices for ξ. As a result, there must exist distinct $\xi, \eta < 2^\omega$ for which the same choices of (A, A', A'', β) are made, that is, in addition to the two expressions above, we also have

$$\bigcup \widetilde{\mathcal{P}_\beta} \subseteq \widetilde{A} \cup \left(\widetilde{A'} \cap \widetilde{A_{\tau_\eta}}\right) \cup \left(\widetilde{A''} \backslash \widetilde{A_{\tau_\eta}}\right), \tag{3.14}$$

$$\bigcup \widetilde{\mathcal{Q}_\beta} \cap \left[\widetilde{A} \cup \left(\widetilde{A'} \cap \widetilde{A_{\tau_\eta}}\right) \cup \left(\widetilde{A''} \backslash \widetilde{A_{\tau_\eta}}\right)\right] = \emptyset. \tag{3.15}$$

Considering unions and/or intersections of (3.12), (3.14) with $\widetilde{A}$, $\widetilde{A'}$ and $\widetilde{A''}$ and bearing in mind that $\widetilde{A}$, $\widetilde{A'}$ and $\widetilde{A''}$ are mutually disjoint, we conclude that

$$\begin{aligned} \bigcup \widetilde{\mathcal{P}_\beta} \cap \widetilde{A} &\subseteq \widetilde{A} \\ \bigcup \widetilde{\mathcal{P}_\beta} \cap \widetilde{A'} &\subseteq \widetilde{A'} \cap \left(\widetilde{A_{\tau_\xi}} \cap \widetilde{A_{\tau_\eta}}\right), \\ \bigcup \widetilde{\mathcal{P}_\beta} \cap \widetilde{A''} &\subseteq \widetilde{A''} \backslash \left(\widetilde{A_{\tau_\xi}} \cap \widetilde{A_{\tau_\eta}}\right). \end{aligned}$$

So, if we define

$$B = \widetilde{A} \cup \left[\widetilde{A'} \cap \left(\widetilde{A_{\tau_\xi}} \cap \widetilde{A_{\tau_\eta}}\right)\right] \cup \left[\widetilde{A''} \backslash \left(\widetilde{A_{\tau_\xi}} \cap \widetilde{A_{\tau_\eta}}\right)\right],$$

then, since $\bigcup \widetilde{\mathcal{P}_\beta} \subseteq \widetilde{A} \cup \widetilde{A'} \cup \widetilde{A''}$, we have

$$\bigcup \widetilde{\mathcal{P}_\beta} \subseteq B.$$

By performing similar manipulations with (3.13) and (3.15) we get

$$\bigcup \widetilde{\mathcal{Q}_\beta} \cap B = \emptyset,$$

which means that the pair $\left(\widetilde{\mathcal{P}_\beta}, \widetilde{\mathcal{Q}_\beta}\right)$ is split by B. However, since $\tau_\xi \cap \tau_\eta \subseteq \sigma_\xi \cap \sigma_\eta$ and so is finite, Claim 3.3.6 in combination with Lemma 3.2.10 and part (iv) of Proposition 3.2.9 imply that

$$\widetilde{A_{\tau_\xi}} \cap \widetilde{A_{\tau_\eta}} = (\widetilde{A})_{\tau_\xi} \cap (\widetilde{A})_{\tau_\eta} = \bigcup_{n \in \tau_\xi \cap \tau_\eta} \widetilde{A_n} \in \widetilde{\mathfrak{A}},$$

which means that $B \in \widetilde{\mathfrak{A}}$ and so, going back to our original $\mathfrak{A}$, the pair $(\mathcal{P}_\beta, \mathcal{Q}_\beta)$ is split in $\mathfrak{A}$ which is a contradiction.

(vi) First of all we show that whenever the splitting $((B_n)_{n\in\tau}, (B_n)_{n\notin\tau})$ is forbidden in $\mathfrak{A}$ for some τ, it remains forbidden in $\mathfrak{A}(\tau)$.

Indeed, suppose that $((B_n)_{n\in\tau}, (B_n)_{n\notin\tau})$ can be split in $\mathfrak{A}(\tau)$. Switching to the dual algebra, and writing $\widetilde{\ }$ for $\widetilde{\ }^{\mathfrak{R}_{\{0,1\}^\alpha}}$, as in the previous part, we can find disjoint $\widetilde{A}, \widetilde{A'}, \widetilde{A''} \in \widetilde{\mathfrak{A}}$ such that

$$\bigcup_{n\in\tau} \widetilde{B_n} \subseteq \widetilde{A} \cup \left(\widetilde{A'} \cap \widetilde{A_\tau}\right) \cup \left(\widetilde{A''} \backslash \widetilde{A_\tau}\right)$$
$$\bigcup_{n\notin\tau} \widetilde{B_n} \cap \left[\widetilde{A} \cup \left(\widetilde{A'} \cap \widetilde{A_\tau}\right) \cup \left(\widetilde{A''} \backslash \widetilde{A_\tau}\right)\right] = \emptyset.$$

Now, recall that we have $\widetilde{A_m} \cap \widetilde{B_n} = \emptyset$ for all m, n. Since $\widetilde{A_\tau}$ as well as each $\widetilde{B_n}$ is clopen in $\mathscr{K}(\mathfrak{R}_{\{0,1\}^\alpha})$, this implies that $\widetilde{A_\tau} \cap \widetilde{B_n} = (\widetilde{A})_\tau \cap \widetilde{B_n} = \emptyset$ for each n. Thus, the last two expressions can be rewritten as

$$\bigcup_{n\in\tau} \widetilde{B_n} \subseteq \widetilde{A} \cup \widetilde{A''}$$
$$\bigcup_{n\notin\tau} \widetilde{B_n} \cap \left[\widetilde{A} \cup \widetilde{A''}\right] = \emptyset,$$

and so $\left((\widetilde{B_n})_{n\in\tau}, (\widetilde{B_n})_{n\notin\tau}\right)$ is split in $\widetilde{\mathfrak{A}}$. Equivalently, $((B_n)_{n\in\tau}, (B_n)_{n\notin\tau})$ is split in $\mathfrak{A}$.

Suppose now that τ, τ' are distinct subsets of σ such that $((B_n)_{n\in\tau}, (B_n)_{n\notin\tau})$ is separated by $A(\tau) \in \mathfrak{A}$ whilst $((B_n)_{n\in\tau'}, (B_n)_{n\notin\tau'})$ is separated by $A(\tau') \in \mathfrak{A}$. Without loss of generality there exists $m \in \tau\backslash\tau'$. Then

$$B_m \subseteq \bigcup_{n\in\tau} B_n \subseteq A(\tau)$$

while

$$B_m \cap A(\tau') \subseteq \left(\bigcup_{n\notin\tau'} B_n\right) \cap A(\tau') = \emptyset.$$

Since $B_m \neq \emptyset$, this means that $A(\tau) \neq A(\tau')$. However, there exist $|\mathfrak{A}| < 2^\omega$ choices for picking $A(\tau) \in \mathfrak{A}$ versus 2^ω choices for picking $\tau \subseteq \sigma$. Thus there exists $\tau \subseteq \sigma$ for which $A(\tau)$ cannot be found or, equivalently, the pair $((B_n)_{n\in\tau}, (B_n)_{n\in\sigma\backslash\tau})$ cannot be split in $\mathfrak{A}$. As shown above, it remains forbidden in $\mathfrak{A}(\tau)$. □

Proposition 3.4.4. *Let $\alpha < 2^\omega$ be an infinite ordinal of the form $\alpha = \beta + 1$ and suppose that $\mathfrak{A}$ is a Boolean algebra with the property that*

(a) $\mathfrak{D}_{\{0,1\}^\beta} \leqslant \mathfrak{A} \leqslant \mathfrak{R}_{\{0,1\}^\beta}$,

(b) $|\mathfrak{A}| < 2^\omega$.

Suppose also that we are given

(c) a disjoint sequence $(X_n)_{n\in\omega} \subseteq \mathfrak{A}$ *with* $X_\omega \in \mathfrak{A}$,

(d) a forbidden splitting $(\mathcal{P}, \mathcal{Q})$ *in* $\mathfrak{A}$.

Define

$$\mathfrak{B} = \left\langle \mathfrak{A}^{[\alpha]}, \mathfrak{D}_{\{0,1\}^\alpha} \right\rangle.$$

Then

(i) $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{B} \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$,

(ii) $|\mathfrak{B}| < 2^\omega$,

(iii) the supremum X_ω *of* $(X_n)_{n\in\omega}$ *is preserved in* $\mathfrak{B}$,

(iv) the forbidden splitting $(\mathcal{P}, \mathcal{Q})$ *is preserved in* $\mathfrak{B}$,

(v) each of $(\mathcal{P} \times \{0\}, \mathcal{Q} \times \{0\})$, $(\mathcal{P} \times \{1\}, \mathcal{Q} \times \{1\})$ *forms a forbidden splitting in* $\mathfrak{B}$.

Proof. (i) By construction, $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{B}$. Furthermore, since $\mathfrak{A}^{[\alpha]} = \{A \times \{0,1\} : A \in \mathfrak{A}\}$ and every element of $\mathfrak{A}$ is a regular open set, we have $\mathfrak{A}^{[\alpha]} \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$. We also have $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$ implying that $\mathfrak{A} \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$.

(ii) The result is immediate as $|\mathfrak{B}| = \max\{|\mathfrak{A}|, |\mathfrak{D}_{\{0,1\}^\alpha}|\} < 2^\omega$.

(iii) Note that

$$\begin{aligned}\mathfrak{B} \ni (X_\omega)^{[\alpha]} = X_\omega \times \{0,1\} &= \operatorname{int}\left(\overline{\bigcup X_n}\right) \times \{0,1\} \\ &= \operatorname{int}\left(\overline{\bigcup X_n \times \{0,1\}}\right) \\ &= \operatorname{int}\left(\overline{\bigcup X_n^{[\alpha]}}\right),\end{aligned}$$

and so the result follows from Lemma 3.2.6.

(iv) First of all note that

$$\mathfrak{D}_{\{0,1\}^\alpha} = \left\langle \mathfrak{D}^{[\alpha]}_{\{0,1\}^\beta}, D_\alpha \right\rangle,$$

where

$$D_\alpha = \left(p^\alpha_\beta\right)^{-1}(\{0\}) = \prod_{\gamma<\beta} \{0,1\} \times \{0\},$$

and thus

$$\mathfrak{B} = \left\langle \mathfrak{A}^{[\alpha]}, D_\alpha \right\rangle.$$

Informally, since D_α is determined by its α^{th} coordinate only while $\mathfrak{A}^{[\alpha]}$, $\mathcal{P}$ and $\mathcal{Q}$ are all determined by their first $\beta < \alpha$ coordinates, an element splitting $\left(\mathcal{P}^{[\alpha]}, \mathcal{Q}^{[\alpha]}\right)$ would have to be of the form $A \times \{0,1\}$ with $A \in \mathfrak{A}$. This would, however, imply that $(\mathcal{P}, \mathcal{Q})$ can be split in $\mathfrak{A}$.

More formally, suppose that $(\mathcal{P} \times \{0,1\}, \mathcal{Q} \times \{0,1\})$ is split by an element of $\mathfrak{B}$. By Lemma 3.2.3, there exist disjoint $A, A', A'' \in \mathfrak{A}$ such that for all $X \in \mathcal{P} \cup \mathcal{Q}$ we have

$$\begin{aligned}(X \times \{0,1\}) \quad &\wedge \quad \Big((A \times \{0,1\}) \vee (A' \times \{0,1\} \wedge D_\alpha) \vee (A'' \times \{0,1\} \wedge \neg D_\alpha)\Big) \\ &= \begin{cases} X \times \{0,1\} & \text{if } X \in \mathcal{P} \\ \emptyset & \text{if } X \in \mathcal{Q}. \end{cases}\end{aligned}$$

But $(A' \times \{0,1\}) \wedge D_\alpha = (A' \times \{0,1\}) \cap D_\alpha = A' \times \{0\}$ while $(A'' \times \{0,1\}) \wedge \neg D_\alpha = A'' \times \{1\}$ which means that the last expression may be rewritten as follows

$$\begin{aligned}(X \times \{0,1\}) \quad &\wedge \quad \Big((A \times \{0,1\}) \vee (A' \times \{0\}) \vee (A'' \times \{1\})\Big) \\ &= \begin{cases} X \times \{0,1\} & \text{if } X \in \mathcal{P} \\ \emptyset & \text{if } X \in \mathcal{Q}. \end{cases}\end{aligned}$$

Let us perform a few more manipulations. Suppose that $X \in \mathcal{P}$. Then

$$\begin{aligned} X \times \{0,1\} &= (X \times \{0,1\}) \wedge \Big((A \times \{0,1\}) \vee (A' \times \{0\}) \vee (A'' \times \{1\})\Big) \\ &\leqslant X \times \{0,1\} \wedge \Big((A \times \{0,1\}) \vee (A' \times \{0,1\}) \vee (A'' \times \{0,1\})\Big) \\ &= \Big[X \wedge (A \vee A' \vee A'')\Big] \times \{0,1\} \\ &\leqslant X \times \{0,1\}, \end{aligned}$$

which implies that

$$X \wedge (A \vee A' \vee A'') = X \quad \forall X \in \mathcal{P}. \tag{3.16}$$

Similarly, if $X \in \mathcal{Q}$ then

$$
\begin{aligned}
\emptyset &= X \times \{0,1\} \wedge \Big((A \times \{0,1\}) \vee (A' \times \{0\}) \vee (A'' \times \{1\}) \Big) \\
&\geqslant (X \times \{0,1\} \wedge A \times \{0,1\}) \vee (X \times \{0\} \wedge A' \times \{0\}) \vee (X \times \{1\} \wedge A'' \times \{1\}) \\
&= \Big((X \wedge A) \times \{0,1\} \Big) \vee \Big((X \wedge A') \times \{0\} \Big) \vee \Big((X \wedge A'') \times \{1\} \Big)
\end{aligned}
$$

However, each of $X \wedge A$, $X \wedge A'$ and $X \wedge A''$ is an open subset of $\{0,1\}^\beta$ and so the last equation implies that

$$X \wedge A = X \wedge A' = X \wedge A'' = \emptyset.$$

Consequently

$$X \wedge (A \vee A' \vee A'') = \emptyset \quad \forall X \in \mathcal{Q}, \tag{3.17}$$

which, combined with (3.16), gives the required contradiction.

The proof of part (v) is precisely the same. $\square$

Proposition 3.4.5. *Let $\alpha \leqslant 2^\omega$ be a limit ordinal and suppose we have a transfinite sequence $(\mathfrak{A}_\beta)_{\omega \leqslant \beta < \alpha}$ of Boolean algebras such that*

(a) $\mathfrak{D}_{\{0,1\}^\beta} \leqslant \mathfrak{A}_\beta \leqslant \mathfrak{R}_{\{0,1\}^\beta}$ whenever $\omega \leqslant \beta < \alpha$,

(b) $|\mathfrak{A}_\beta| < 2^\omega$ whenever $\omega \leqslant \beta < \alpha$,

(c) $\mathfrak{A}_\gamma^{[\beta]} \leqslant \mathfrak{A}_\beta$ whenever $\omega \leqslant \gamma \leqslant \beta < \alpha$.

Fix now some β_0 with $\omega \leqslant \beta_0 < \alpha$ and suppose that, in addition to the above, we are given

(d) a disjoint sequence $(X_n)_{n \in \omega} \subseteq \mathfrak{A}_{\beta_0}$ with $X_\omega \in \mathfrak{A}_{\beta_0}$,

(e) a forbidden splitting $(\mathcal{P}, \mathcal{Q}) \subseteq \mathfrak{A}_{\beta_0} \times \mathfrak{A}_{\beta_0}$.

Define

$$\mathfrak{A}_\alpha = \bigcup_{\omega \leqslant \beta < \alpha} \mathfrak{A}_\beta^{[\alpha]}.$$

Then

(i) $\mathfrak{A}_\alpha$ is a Boolean algebra with $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{A}_\alpha \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$,

(ii) if $\alpha < 2^\omega$, then $|\mathfrak{A}_\alpha| < 2^\omega$,

(iii) the supremum X_ω of $(X_n)_{n\in\omega}$ is preserved in $\mathfrak{A}_\beta$ for each β with $\beta_0 \leqslant \beta \leqslant \alpha$,

(iv) if the forbidden splitting $(\mathcal{P},\mathcal{Q})$ is preserved in $\mathfrak{A}_\beta$ for each β with $\beta_0 \leqslant \beta < \alpha$, then it is preserved in $\mathfrak{A}_\alpha$.

Proof. (i) Since $\mathfrak{A}_\gamma^{[\beta]} \leqslant \mathfrak{A}_\beta$ for all $\gamma \leqslant \beta < \alpha$, it follows that $\mathfrak{A}_\alpha$ is a Boolean algebra.

Furthermore, note that any clopen subset A of $\{0,1\}^\alpha$ is defined by finitely many coordinates and so, since α is a limit ordinal, there exists $\beta < \alpha$ with

$$\left(\pi^\alpha_\beta\right)^{-1}\left(\pi^\alpha_\beta(A)\right) = A.$$

Now, the set $\pi^\alpha_\beta(A)$ is still defined by finitely many coordinates. Thus $\pi^\alpha_\beta(A) \in \mathfrak{A}_\beta$ and $A = (\pi^\alpha_\beta(A))^{[\alpha]} \in \mathfrak{A}_\alpha$. This shows that $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{A}_\alpha$.

Furthermore, note that since each of $\mathfrak{R}_{\{0,1\}^\beta}$ consists of regular open sets, so does $\mathfrak{R}^{[\alpha]}_{\{0,1\}^\beta}$ which means that $\mathfrak{A}_\alpha \leqslant \bigcup_{\beta<\alpha} \mathfrak{R}^{[\alpha]}_{\{0,1\}^\beta} \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$.

(ii) This part follows immediately from the assumption (b).

(iii) By the same calculations as in part (iii) of Proposition 3.4.4, it can be shown that

$$\mathfrak{A}_\beta \ni (X_\omega)^{[\beta]} = \operatorname{int}\left(\overline{\bigcup (X_n^{[\beta]})}\right)$$

whenever $\beta_0 \leqslant \beta \leqslant \alpha$, and so the result follows from Lemma 3.2.6.

(iv) Suppose that $(\mathcal{P},\mathcal{Q})$ is not preserved in $\mathfrak{A}_\alpha$. Then we can find $\beta < \alpha$ and $A \in \mathfrak{A}_\beta$ such that $\left(\mathcal{P}^{[\alpha]},\mathcal{Q}^{[\alpha]}\right)$ is split by $A^{[\alpha]}$. If $\beta < \beta_0$, then $A^{[\beta_0]}$ is an element of $\mathfrak{A}_{\beta_0}$ which separates $(\mathcal{P},\mathcal{Q})$, leading to a contradiction. Otherwise $\left(\mathcal{P}^{[\beta]},\mathcal{Q}^{[\beta]}\right)$ is split in $\mathfrak{A}_\beta$ by A which leads to another contradiction. □

3.5 Construction of K

Our space K is obtained as the Stone space of a Boolean algebra $\mathfrak{A}$ which, in turn, is constructed by transfinite induction.

Notation

Fix a surjection $s\colon \{\alpha : \alpha \text{ is a successor ordinal with } \omega < \alpha < 2^\omega\} \to (2^\omega \setminus \bigcup \omega) \times 2^\omega$ such that if $s(\alpha) = (\eta,\zeta)$, then $\eta < \alpha$.

Inductive construction

We are going to construct

(i) a transfinite sequence $(\mathfrak{A}_\alpha)_{\omega\leqslant\alpha<2^\omega}$ of Boolean algebras such that for each α

(i.a) $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{A}_\alpha \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$,

(i.b) $|\mathfrak{A}_\alpha| < 2^\omega$,

(i.c) $\mathfrak{A}_\beta^{[\alpha]} \leqslant \mathfrak{A}_\alpha$ whenever $\omega \leqslant \beta < \alpha$,

(i.d) if $\omega \leqslant \beta < \alpha$ and (X_n) is a disjoint sequence in $\mathfrak{A}_\beta$ with $X_\omega \in \mathfrak{A}_\beta$, then the supremum X_ω is preserved in $\mathfrak{A}_\alpha$,

(i.e) we fix an enumeration of all the quadruples $\left((A_n(\alpha,\zeta))_{n\in\omega}, (B_n(\alpha,\zeta))_{n\in\omega}, (C_n(\alpha,\zeta))_{n\in\omega}, (D_n(\alpha,\zeta))_{n\in\omega}\right)$ $(\zeta < 2^\omega)$ with the property that

- each of the four sequences is disjoint and consists of elements of $\mathfrak{A}_\alpha$,
- all $A_n(\alpha,\zeta)$ and $B_n(\alpha,\zeta)$ are nonempty,
- $A_m(\alpha,\zeta) \cap B_n(\alpha,\zeta) = \emptyset$ for all $m,n \in \omega$ and $\zeta < 2^\omega$,

(i.f) using the enumeration from (i.e), if $\alpha = \beta + 1$, there exists $\tau \subseteq \omega$ such that

- $\mathfrak{A}_\alpha = \left\langle \mathfrak{A}_\beta(\tau)^{[\alpha]}, \mathfrak{D}_{\{0,1\}^\alpha} \right\rangle$, where $\mathfrak{A}_\beta(\tau) = \left\langle \mathfrak{A}_\beta, \left(A(s(\alpha))^{[\beta]}\right)_\tau \right\rangle$,

(i.g) if α is a limit ordinal, then

- $\mathfrak{A}_\alpha = \bigcup_{\omega\leqslant\beta<\alpha} \mathfrak{A}_\beta^{[\alpha]}$;

(ii) arrays $(\mathcal{P}_{\alpha,i}, \mathcal{Q}_{\alpha,i})$ with $\omega \leqslant \alpha < 2^\omega$ and $i \in \{0,1,10,11\}$ with the property that

(ii.a) if $\alpha = \beta + 1$ with $\omega \leqslant \beta$, then

- $(\mathcal{P}_{\alpha,0}, \mathcal{Q}_{\alpha,0}) = \left((B_n(s(\alpha))^{[\alpha]})_{n\in\tau}, (B_n(s(\alpha))^{[\alpha]})_{n\notin\tau}\right)$ with τ as in (i.f),
- $(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1}) = \left((C_n(s(\alpha))^{[\alpha]})_{n\in\omega}, (D_n(s(\alpha))^{[\alpha]})_{n\in\omega}\right)$ if this pair cannot be split in $\mathfrak{A}_\alpha$, or $(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1}) = (\emptyset,\emptyset)$ otherwise,
- $(\mathcal{P}_{\alpha,10}, \mathcal{Q}_{\alpha,10}) = (\mathcal{P}_{\beta,1} \times \{0\}, \mathcal{Q}_{\beta,1} \times \{0\})$,
- $(\mathcal{P}_{\alpha,11}, \mathcal{Q}_{\alpha,11}) = (\mathcal{P}_{\beta,1} \times \{1\}, \mathcal{Q}_{\beta,1} \times \{1\})$,

(ii.b) if α is a limit ordinal, then

- $(\mathcal{P}_{\alpha,i}, \mathcal{Q}_{\alpha,i}) = (\emptyset,\emptyset)$ for each i, (ii.c) $(\mathcal{P}_{\alpha,i}, \mathcal{Q}_{\alpha,i})$ forms a forbidden splitting in $\mathfrak{A}_\alpha$ unless it is equal to $(\emptyset, \emptyset)$,

(ii.d) if $\omega \leqslant \beta < \alpha$ and $(\mathcal{P}_{\beta,i}, \mathcal{Q}_{\beta,i})$ is forbidden in $\mathfrak{A}_\beta$, it remains forbidden in $\mathfrak{A}_\alpha$.

Note 3.5.1. Existence of the enumeration in (i.e) follows from (i.b) and does not need to be taken care of separately.

Base case

Define

$$\mathfrak{A}_\omega = \mathfrak{D}_{\{0,1\}^\omega},$$

and for each $i \in \{0, 1, 10, 11\}$ set

$$(\mathcal{P}_{\omega,i}, \mathcal{Q}_{\omega,i}) = (\emptyset, \emptyset).$$

Inductive step

Suppose that for some α with $\omega \leqslant \alpha < 2^\omega$ we have constructed $(\mathfrak{A}_\beta)_{\beta<\alpha}$ and $((\mathcal{P}_{\beta,i}, \mathcal{Q}_{\beta,i}))$ $(\beta < \alpha,\ i \in \{0, 1, 10, 11\})$ satisfying conditions (i.a)–(ii.d).

Case 1. α is a successor ordinal

Suppose that $\alpha = \beta + 1$ for some ordinal $\beta < 2^\omega$.

To construct $\mathfrak{A}_\alpha$, we start with taking care of property (K$'$). Suppose that $s(\alpha) = (\eta, \zeta)$. Then $\omega \leqslant \eta \leqslant \beta$ which means that η^{th} stage of our construction has already been completed. Using the enumeration fixed in (i.e), for each $n \in \omega$ we define

$$A_n = A_n(\eta, \zeta)^{[\beta]}, \quad B_n = B_n(\eta, \zeta)^{[\beta]},$$

$$C_n = C_n(\eta, \zeta)^{[\beta]}, \quad D_n = D_n(\eta, \zeta)^{[\beta]}.$$

Then each of (A_n), (B_n), (C_n), (D_n) is a disjoint sequence consisting of nonempty elements of $\mathfrak{A}_\beta$ and, in addition, $A_m \cap B_n = \emptyset$ for all m, n.

At this stage there are at most $4\beta < 2^\omega$ splittings which are forbidden in $\mathfrak{A}_\beta$ and need to be preserved. Hence, by Proposition 3.4.3, there exists an infinite $\tau \subseteq \omega$ such that

- the algebra $\mathfrak{A}_\beta(\tau) = \langle \mathfrak{A}_\beta, A_\tau \rangle$ satisfies the conditions (i.a)–(i.c),
- if (X_n) is a disjoint sequence in $\mathfrak{A}_\beta$ with $X_\omega \in \mathfrak{A}_\beta$, then the supremum X_ω is preserved in $\mathfrak{A}_\beta(\tau)$,

- a supremum of $(A_n)_{n\in\tau}$ exists in $\mathfrak{A}_\beta(\tau)$ and is equal to A_τ,
- if, for some $\gamma \leqslant \beta$ and $i \in \{0,1,10,11\}$, the splitting $\left(\mathcal{P}^{[\beta]}_{\gamma,i}, \mathcal{Q}^{[\beta]}_{\gamma,i}\right)$ is forbidden in $\mathfrak{A}_\beta$, it remains forbidden in $\mathfrak{A}_\beta(\tau)$,
- the pair $((B_n)_{n\in\tau}, (B_n)_{n\notin\tau})$ cannot be split in $\mathfrak{A}_\beta(\tau)$.

With τ as above we define

$$\mathfrak{A}_\alpha = \left\langle \mathfrak{A}_\beta(\tau)^{[\alpha]}, \mathfrak{D}_{\{0,1\}^\alpha} \right\rangle .$$

Appealing to Proposition 3.4.4,

- $\mathfrak{A}_\alpha$ satisfies the conditions (i.a)–(i.c) and (i.e)–(i.f),
- if (X_n) is a disjoint sequence in $\mathfrak{A}_\beta(\tau)$ with $X_\omega \in \mathfrak{A}_\beta(\tau)$, then the supremum X_ω is preserved in $\mathfrak{A}_\alpha$,
- if the splitting $(\mathcal{P}, \mathcal{Q})$ is forbidden in $\mathfrak{A}_\beta(\tau)$, it remains forbidden in $\mathfrak{A}_\alpha$,
- if the splitting $(\mathcal{P}, \mathcal{Q})$ is forbidden in $\mathfrak{A}_\beta(\tau)$, then both $(\mathcal{P}\times\{0\}, \mathcal{Q}\times\{0\})$ and $(\mathcal{P}\times\{1\}, \mathcal{Q}\times\{1\})$ are forbidden in $\mathfrak{A}_\alpha$.

To finish the construction, we define

$$(\mathcal{P}_{\alpha,0}, \mathcal{Q}_{\alpha,0}) = \left((B^{[\alpha]}_n)_{n\in\tau}, (B^{[\alpha]}_n)_{n\notin\tau}\right),$$

$$(\mathcal{P}_{\alpha,10}, \mathcal{Q}_{\alpha,10}) = (\mathcal{P}_{\beta,1}\times\{0\}, \mathcal{Q}_{\beta,1}\times\{0\}),$$

$$(\mathcal{P}_{\alpha,11}, \mathcal{Q}_{\alpha,11}) = (\mathcal{P}_{\beta,1}\times\{1\}, \mathcal{Q}_{\beta,1}\times\{1\})$$

and

$$(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1}) = \begin{cases} \left((C^{[\alpha]}_n)_{n\in\omega}, (D^{[\alpha]}_n)_{n\in\omega}\right) & \text{if } \left((C^{[\alpha]}_n)_{n\in\omega}, (D^{[\alpha]}_n)_{n\in\omega}\right) \text{ cannot be split in } \mathfrak{A}_\alpha \\ (\emptyset,\emptyset) & \text{otherwise.} \end{cases}$$

Then the splittings $(\mathcal{P}_{\alpha,i}, \mathcal{Q}_{\alpha,i})$ satisfy the conditions (ii.a)–(ii.c). Furthermore, if for some $\gamma < \alpha$ and $i \in \{0,1,10,11\}$, the splitting $(\mathcal{P}_{\gamma,i}, \mathcal{Q}_{\gamma,i})$ is forbidden in $\mathfrak{A}_\gamma$, by the inductive assumption, it remains forbidden in $\mathfrak{A}_\beta$. Thus it remains forbidden in $\mathfrak{A}_\beta(\tau)$ and hence also in $\mathfrak{A}_\alpha$. Consequently, $\mathfrak{A}_\alpha$ satisfies the condition (ii.d).

Finally, suppose that for some $\gamma < \alpha$ we have a disjoint sequence $(X_n) \subseteq \mathfrak{A}_\gamma$ with $X_\omega \in \mathfrak{A}_\gamma$. By the inductive assumption, the supremum X_ω is preserved in $\mathfrak{A}_\beta$. Thus it is preserved in $\mathfrak{A}_\beta(\tau)$ and hence in $\mathfrak{A}_\alpha$. Consequently, $\mathfrak{A}_\alpha$ satisfies the condition (i.d).

Case 2. α is a limit ordinal

In this case we define

$$\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta^{[\alpha]}$$

and for each $i \in \{0, 1, 10, 11\}$ set

$$(\mathcal{P}_{\alpha,i}, \mathcal{Q}_{\alpha,i}) = (\emptyset, \emptyset)\,.$$

Proposition 3.4.5 guarantees that the conditions (i.a)–(ii.d) are satisfied at this stage.

Construction of $\mathfrak{A}$ and K

Once the construction of $(\mathfrak{A}_\alpha)_{\omega\leqslant\alpha<2^\omega}$ has been completed, we define

$$\mathfrak{A} = \bigcup_{\omega\leqslant\alpha<2^\omega} \mathfrak{A}_\alpha^{[2^\omega]}$$

and set

$$K = \mathscr{K}(\mathfrak{A}).$$

3.6 Properties of K

Proposition 3.6.1 (P1)**.** *K is separable.*

Proof. By Corollary 3.2.16, it is enough to show that $\mathfrak{A} \leqslant \mathfrak{R}_{\{0,1\}^{2^\omega}}$ and this was done in part (i) of Proposition 3.4.5. □

Proposition 3.6.2 (P2)**.** *K is weakly Koszmider.*

Proof. By Theorem 3.3.2, it is sufficient to show that $\mathfrak{A}$ has property (K′).

Let $(A_n), (B_n)$ be disjoint sequences of nonempty element of $\mathfrak{A}$ with $A_m \wedge B_n = \emptyset$ for all m, n. By construction, for each n there exist infinite $\alpha_n, \beta_n < 2^\omega$, $\underline{A}_n \in \mathfrak{A}_{\alpha_n}$, $\underline{B}_n \in \mathfrak{A}_{\beta_n}$ such that

$$A_n = \underline{A}_n^{[2^\omega]}, \quad B_n = \underline{B}_n^{[2^\omega]}.$$

By König's Lemma [Kun80, p. 34, Lemma 10.40], there exists $\eta < 2^\omega$ such that $\alpha_n, \beta_n < \eta$ for all n. Then for each n we have

$$A_n = \underline{A}_n^{[2^\omega]} = \left(\pi^{2^\omega}_{\alpha_n}\right)^{-1}(\underline{A}_n) = \left(\pi^{2^\omega}_{\eta}\right)^{-1}\left(\left(\pi^{\eta}_{\alpha_n}\right)^{-1}(\underline{A}_n)\right) = \left(\pi^{2^\omega}_{\eta}\right)^{-1}(\underline{A}_n^{[\eta]})$$

and similarly

$$B_n = \left(\pi^{2^\omega}_\eta\right)^{-1}(\underline{B}^{[\eta]}_n).$$

Thus $(\underline{A}^{[\eta]}_n)$, $(\underline{B}^{[\eta]}_n)$ are disjoint sequences of nonempty elements of $\mathfrak{A}_\eta$ with $\underline{A}^{[\eta]}_m \wedge \underline{B}^{[\eta]}_n = \emptyset$ for all m, n. Using the enumeration fixed in (i.e), we can find $\zeta < 2^\omega$ such that for each n

$$A^{[\eta]}_n = A_n(\eta,\zeta), \quad B^{[\eta]}_n = B_n(\eta,\zeta).$$

Take now any infinite successor ordinal $\alpha < 2^\omega$ with $s(\alpha) = (\eta,\zeta)$. By the construction algorithm, there exists $\tau \subseteq \omega$ such that

- a supremum of $\left((\underline{A}^{[\eta]}_n)^{[\alpha]}\right)_{n\in\tau}$ exists in $\mathfrak{A}_\alpha$ and is equal to $\left((\underline{A}^{[\eta]})^{[\alpha]}\right)_\tau$,
- the pair $\left(\left((\underline{B}^{[\eta]}_n)^{[\alpha]}\right)_{n\in\tau}, \left((\underline{B}^{[\eta]}_n)^{[\alpha]}\right)_{n\notin\tau}\right)$ forms the splitting $(\mathcal{P}_{\alpha,0},\mathcal{Q}_{\alpha,0})$ and is forbidden in $\mathfrak{A}_\alpha$.

By construction, the supremum $\left((\underline{A}^{[\eta]})^{[\alpha]}\right)_\tau$ is preserved in all $\mathfrak{A}_{\alpha'}$ with $\alpha \leqslant \alpha' < 2^\omega$ which, by Proposition 3.4.5, means that

$$\begin{aligned}
\mathfrak{A} \ni \left(\left((\underline{A}^{[\eta]})^{[\alpha]}\right)_\tau\right)^{[2^\omega]} &= \left(\left((\underline{A}^{[\eta]})^{[\alpha]}\right)^{[2^\omega]}\right)_\tau \\
&= \operatorname{int}\left(\overline{\bigcup_{n\in\tau}(\pi^{2^\omega}_\alpha)^{-1}\left((\pi^\alpha_\eta)^{-1}\left((\pi^\eta_{\alpha_n})^{-1}(\underline{A}_n)\right)\right)}\right) \\
&= \operatorname{int}\left(\overline{\bigcup_{n\in\tau}\left(\pi^{2^\omega}_{\alpha_n}\right)^{-1}(\underline{A}_n)}\right) \\
&= \operatorname{int}\left(\overline{\bigcup_{n\in\tau}\underline{A}^{[2^\omega]}_n}\right) \\
&= \operatorname{int}\left(\overline{\bigcup_{n\in\tau}A_n}\right) \\
&= A_\tau.
\end{aligned}$$

Similarly, the forbidden splitting $(\mathcal{P}_{\alpha,0},\mathcal{Q}_{\alpha,0})$ is preserved in $\mathfrak{A}_{\alpha'}$ whenever $\alpha \leqslant \alpha' < 2^\omega$ which, by Proposition 3.4.5, means that the pair

$$\begin{aligned}
\left(\mathcal{P}^{[2^\omega]}_{\alpha,0},\mathcal{Q}^{[2^\omega]}_{\alpha,0}\right) &= \left(\left(\left((\underline{B}^{[\eta]}_n)^{[\alpha]}\right)^{[2^\omega]}\right)_{n\in\tau}, \left(\left((\underline{B}^{[\eta]}_n)^{[\alpha]}\right)^{[2^\omega]}\right)_{n\in\tau}\right) \\
&= \left(\left(\underline{B}^{[2^\omega]}_n\right)_{n\in\tau}, \left(\underline{B}^{[2^\omega]}_n\right)_{n\notin\tau}\right) \\
&= ((B_n)_{n\in\tau},(B_n)_{n\notin\tau})
\end{aligned}$$

remains forbidden in $\mathfrak{A}$.

The above two conclusions show us that $\mathfrak{A}$ has property (K′). $\square$

Proposition 3.6.3 (P3)**.** *K has no open butterflies.*

We first prove the following general result.

Lemma 3.6.4. *Let K be a separable topological space. Fix a basis $\mathscr{B}$ of K and let $U \subseteq K$ be open. There exists a disjoint sequence $(U_n) \subseteq \mathscr{B}$ such that*

$$\overline{U} = \overline{\bigcup U_n}.$$

Proof. Let $D = \{d_n : n \in \omega\}$ be a dense subset of K. Note that

$$\overline{U} = \overline{U \cap D}.$$

Indeed, if $x \in \overline{U}$ and $A \ni x$ is open then $U \cap A$ is a nonempty open subset of K. Thus $\emptyset \neq (U \cap A) \cap D = (U \cap D) \cap A$ and so $x \in \overline{U \cap D}$. The reverse inclusion is obvious.

We now construct a sequence $(U_n) \subseteq \mathscr{B}$ using the following inductive algorithm:

Base Case. If $d_0 \in U$, we define U_0 to be any element of $\mathscr{B}$ with $d_0 \in U_0 \subseteq U$. Otherwise we set $U_0 = \emptyset$.

Inductive Step. Suppose that $U_0, \ldots, U_{n-1}$ have been constructed. We define

$$U_n = \begin{cases} \text{an element of } \mathscr{B} \text{ s.t. } d_n \in U_n \subseteq U \backslash \overline{\bigcup_{0 \leqslant j < n} U_j} & \text{if } d_n \in U \backslash \overline{\bigcup_{0 \leqslant j < n} U_j} \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly (U_n) is a disjoint sequence and $\bigcup U_n \subseteq U$. Furthermore, if $x \in \overline{U \cap D}$ and $A \ni x$ is open, there exists n with $d_n \in A \cap U$. If $d_n \in \overline{\bigcup_{0 \leqslant j < n} U_j}$, then $A \cap U_j \neq \emptyset$ for some j with $0 \leqslant j < n$. Otherwise $d_n \in U_n$ and $A \cap U_n \neq \emptyset$. In either case $A \cap \bigcup U_j \neq \emptyset$ which means that $x \in \overline{\bigcup U_j}$. Thus

$$\overline{\bigcup U_j} \subseteq \overline{U} = \overline{U \cap D} \subseteq \overline{\bigcup U_j}$$

giving the required result. $\square$

Proof of Proposition 3.6.3. Fix a basis $\mathscr{B}$ of K consisting of clopen sets and suppose that V and W are open subsets of K with

$$\overline{V} \cap \overline{W} \neq \emptyset.$$

By Lemma 3.6.4, there exist disjoint clopen $(V_n), (W_n) \subseteq K$ such that $\overline{\bigcup V_n} = V$ and $\overline{\bigcup W_n} = W$ and hence

$$\overline{\bigcup V_n} \cap \overline{\bigcup W_n} \neq \emptyset. \tag{3.18}$$

Since $K = \mathscr{K}(\mathfrak{A})$, by the Stone Representation Theorem, $\mathfrak{A}$ is isomorphic (via an isomorphism $\widetilde{\ }$, as usual) to $\mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$. In particular, since $(V_n), (W_n) \subseteq \mathfrak{D}_{\mathscr{K}(\mathfrak{A})}$, there exist $(C_n), (D_n) \subseteq \mathfrak{A}$ such that for each n we have

$$\widetilde{C_n} = V_n, \quad \widetilde{D_n} = W_n.$$

By part (iii) of Proposition 3.2.9, disjointness of (V_n) and (W_n) implies that both (C_n) and (D_n) are disjoint, while (3.18), Lemma 3.2.10 and part (v) of Proposition 3.2.9 guarantee that $((C_n), (D_n))$ is a forbidden splitting in $\mathfrak{A}$.

We now proceed exactly as in the previous part. For each n there exist infinite $\alpha_n, \beta_n < 2^\omega$, $\underline{C}_n \in \mathfrak{A}_{\alpha_n}$ and $\underline{D}_n \in \mathfrak{A}_{\beta_n}$ such that

$$C_n = \underline{C}_n^{[2^\omega]}, \quad D_n = \underline{D}_n^{[2^\omega]}.$$

By König's Lemma [Kun80, p. 34, Lemma 10.40], there exists $\eta < 2^\omega$ such that for all n we have $\alpha_n, \beta_n < \eta$. Then $(\underline{C}_n^{[\eta]})$, $(\underline{D}_n^{[\eta]})$ are disjoint sequences in $\mathfrak{A}_\eta$. Consequently, using the enumeration fixed in (i.e), we can find $\zeta < 2^\omega$ such that for all n

$$\underline{C}_n^{[\eta]} = C_n(\eta, \zeta), \quad \underline{D}_n^{[\eta]} = D_n(\eta, \zeta).$$

Let $\alpha < 2^\omega$ be an infinite successor ordinal with $s(\alpha) = (\eta, \zeta)$. Then $\alpha_n, \beta_n < \eta < \alpha$ for each n and so

$$C_n = \left(\pi_\alpha^{2^\omega}\right)^{-1} (\underline{C}_n^{[\alpha]}), \quad D_n = \left(\pi_\alpha^{2^\omega}\right)^{-1} (\underline{D}_n^{[\alpha]})$$

which means that the pair $\left((\underline{C}_n^{[\alpha]}), (\underline{D}_n^{[\alpha]})\right)$ cannot be split in $\mathfrak{A}_\alpha$ and so forms the forbidden splitting $(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1})$.

By construction, for each $i \in \{0,1\}$ the pair $\left((C_n^{[\alpha]} \times \{i\}), (D_n^{[\alpha]} \times \{i\})\right)$ forms a forbidden splitting $(\mathcal{P}_{\alpha+1,1i}, \mathcal{Q}_{\alpha+1,1i})$ in $\mathfrak{A}_{\alpha+1}$ which is then preserved in $\mathfrak{A}_{\alpha'}$ for all $\alpha+1 \leqslant \alpha' < 2^\omega$. By Proposition 3.4.5, it remains forbidden in $\mathfrak{A}$.

Going back to $\widetilde{\mathfrak{A}} = \mathfrak{D}_K$, this is equivalent to saying that for each $i \in \{0,1\}$ there exists

$$x_i \in \overline{\bigcup \widetilde{(\underline{C}_n^{[\alpha]} \times \{i\})^{[2^\omega]}}} \cap \overline{\bigcup \widetilde{(\underline{D}_n^{[\alpha]} \times \{i\})^{[2^\omega]}}},$$

and, of course, since

$$\begin{aligned}
\overline{\bigcup(\widetilde{\underline{C}_n^{[\alpha]} \times \{i\})^{[2^\omega]}} \cap \overline{\bigcup(\widetilde{\underline{D}_n^{[\alpha]} \times \{i\})^{[2^\omega]}} &\subseteq \overline{\bigcup(\widetilde{\underline{C}_n^{[\alpha]} \times \{0,1\})^{[2^\omega]}} \cap \overline{\bigcup(\widetilde{\underline{D}_n^{[\alpha]} \times \{0,1\})^{[2^\omega]}} \\
&= \overline{\bigcup(\widetilde{\underline{C}_n^{[\alpha+1]})^{[2^\omega]}} \cap \overline{\bigcup(\widetilde{\underline{D}_n^{[\alpha+1]})^{[2^\omega]}} \\
&= \overline{\bigcup(\widetilde{\underline{C}_n^{[2^\omega]}})} \cap \overline{\bigcup(\widetilde{\underline{D}_n^{[2^\omega]}})} \\
&= \overline{\bigcup \widetilde{C}_n} \cap \overline{\bigcup \widetilde{D}_n} \\
&= \overline{\bigcup V_n} \cap \overline{\bigcup W_n} \\
&= \overline{V} \cap \overline{W},
\end{aligned}$$

we have

$$x_i \in \overline{V} \cap \overline{W}.$$

Finally, note that the pair $\left((\underline{C}_n^{[\alpha]} \times \{0\})^{[2^\omega]}, (\underline{C}_n^{[\alpha]} \times \{1\})^{[2^\omega]}\right)$ is split in $\mathfrak{A}$ by $\left(p_{\alpha+1}^{2^\omega}\right)^{-1}(\{0\})$ which means that $\left(\widetilde{(\underline{C}_n^{[\alpha]} \times \{0\})^{[2^\omega]}}, \widetilde{(\underline{C}_n^{[\alpha]} \times \{1\})^{[2^\omega]}}\right)$ is split in $\widetilde{\mathfrak{A}}$ and so

$$\overline{\bigcup(\widetilde{\underline{C}_n^{[\alpha]} \times \{0\})^{[2^\omega]}} \cap \overline{(\widetilde{\underline{C}_n^{[\alpha]} \times \{1\})^{[2^\omega]}} = \emptyset.$$

Consequently

$$x_0 \neq x_1,$$

and so $\overline{V} \cap \overline{W}$ contains at least two points. □

Chapter 4

Construction of a separable connected Koszmider space

4.1 Introduction

4.1.1 Description of the construction. Boolean algebras versus topological spaces

In the spirit of the previous chapter, we construct a topological space K such that

(P1) K is separable,

(P2) K is weakly Koszmider,

(P3) K has no open butterflies,

(P4) K is connected.

Using Stone duality, the algorithm from Chapter 3 can be modified in a natural way to allow us to deal with topological spaces instead of Boolean algebras. For convenience, we include a comparison table which contains a summary of (most important) changes we introduced.

All notation and terminology in this table was either described in Chapter 3 or will be defined later. In addition, for aesthetic purposes, we omit the "liftings" notation and so, for example, the statement $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$ should be read as $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta^{[\alpha]}$ etc.

Zero-dimensional construction	Connected construction
(i.a) A transfinite sequence $(\mathfrak{A}_\alpha)_{\alpha<2^\omega}$ of Boolean algebras.	(i.a) A transfinite sequence $(K_\alpha)_{\alpha<2^\omega}$ of topological spaces.
(i.b) $\cdots \leqslant \mathfrak{A}_\alpha \leqslant \mathfrak{A}_{\alpha+1} \leqslant \cdots \leqslant \mathfrak{A}$, or, equivalently, $\mathscr{K}(\mathfrak{A}) \twoheadrightarrow \cdots \twoheadrightarrow \mathscr{K}(\mathfrak{A}_{\alpha+1}) \twoheadrightarrow \mathscr{K}(\mathfrak{A}_\alpha) \twoheadrightarrow \cdots$	(i.b) $K \twoheadrightarrow \cdots \twoheadrightarrow K_{\alpha+1} \twoheadrightarrow K_\alpha \twoheadrightarrow \cdots$
(i.c) $\mathfrak{D}_{\{0,1\}^\alpha} \leqslant \mathfrak{A}_\alpha \leqslant \mathfrak{R}_{\{0,1\}^\alpha}$, or, equivalently, $\mathscr{G}_{\{0,1\}^\alpha} \twoheadrightarrow \mathscr{K}(\mathfrak{A}_\alpha) \twoheadrightarrow \{0,1\}^\alpha$.	(i.c) $\mathscr{G}_{[0,1]^\alpha} \twoheadrightarrow \mathscr{K}(\mathfrak{A}_\alpha) \twoheadrightarrow [0,1]^\alpha$.
(i.d) If $\alpha = \beta + 1$, then $$\mathfrak{A}_\alpha = \left\langle \mathfrak{A}_\beta(\tau) \times \{0,1\}, \mathfrak{D}_{\{0,1\}^\alpha} \right\rangle,$$ where $\mathfrak{A}_\beta(\tau) = \langle \mathfrak{A}_\beta, A_\tau \rangle$ and $A_\tau = \vee_{n\in\tau} A_n$	(i.d) If $\alpha = \beta + 1$, then $$K_\alpha = K_\beta(\tau) \times [0,1],$$ where $K_\beta(\tau)$ is the (strong) extension of K by $(f_n)_{n\in\tau}$. Equivalently (see Proposition 4.2.17), $$\mathscr{C}(K_\alpha) = \overline{\langle \mathscr{C}(K_\beta(\tau)), \mathscr{C}([0,1]) \rangle},$$ where $\mathscr{C}(K_\beta(\tau)) = \overline{\langle \mathscr{C}(K_\beta), f_\tau \rangle}$ and $f_\tau = \vee_{n\in\tau} f_n$.
(i.e) If α is a limit ordinal, then $$\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta.$$	(i.e) If α is a limit ordinal, then $$K_\alpha = \varprojlim_{\beta<\alpha} K_\beta.$$
(i.f) The resulting space is $\mathscr{K}(\mathfrak{A})$ where $$\mathfrak{A} = \bigcup_{\alpha<2^\omega} \mathfrak{A}_\alpha.$$	(i.f) The resulting space is $$K = \varprojlim_{\alpha<2^\omega} K_\alpha.$$
(ii.a) Property (K$'$).	(ii.a) Property (K).
(ii.b) (A_n) is a disjoint sequence of nonempty elements of $\mathfrak{A}$, (B_n) is a disjoint sequence of nonempty elements of $\mathfrak{A}$, $A_m \cap B_n = \emptyset$ for all m, n.	(ii.b) (f_n) is a disjoint sequence in $\mathscr{C}(K)$ with range$(f_n) \subseteq [0,1]$ for each n, (U_n) is a disjoint sequence of nonempty open subsets of K, supp$(f_m) \cap U_n = \emptyset$ for all m, n.
(iii.a) (C_n), (D_n) are disjoint sequences of clopen subsets of $\mathfrak{A}$.	(iii.a) (V_n), (W_n) are disjoint sequences of basic open subsets of K.
(iii.b) If $\alpha = \beta + 1$ and $i \in \{0,1\}$, $$(\mathcal{P}_{\alpha,1i}, \mathcal{Q}_{\alpha,1i}) = (\mathcal{P}_{\beta,1} \times \{i\}, \mathcal{Q}_{\beta,1} \times \{i\}).$$	(iii.b) If $\alpha = \beta + 1$ and $I \in \{[0,1/3], [2/3,1]\}$, $$(\mathcal{P}_{\alpha,I}, \mathcal{Q}_{\alpha,I}) = (\mathcal{P}_{\beta,1} \times I, \mathcal{Q}_{\beta,1} \times I).$$

Note that the space $[0,1]^{2^\omega}$ is separable (see e.g. [Dug66, p.175, Theorem 7.2]) thus, as before, (i.c) implies that so is K.

To get (P2), we introduce property (K) to state which we reformulate the definition of property (K′) making substitutions described in (ii.b). Note that property (K) is similar to property (H) from [Ple04] as well as to the property described in [Kos04, Theorem 5.1]. The main difference is the same as in the zero-dimensional case, namely, we replace a sequence (x_n) of points in K with a sequence (U_n) of nonempty open subsets of K. We show that a space with property (K) is weakly Koszmider. To construct a space with property (K), we use the same idea of adding and preserving suprema of disjoint functions and forbidden splittings as before.

As in the zero-dimensional case, to ensure that K has no open butterflies, instead of adding one pair to the list of forbidden splittings, at each step of our construction we add up to four new pairs.

Finally, since K is constructed as the inverse limit of the spaces K_α, to get (P4), it is sufficient to ensure that each K_α is connected. To do this, we use tools from [Kos04]. Specifically, Koszmider showed that *strong extensions* of topological spaces preserve connectedness and so, as described in (i.d), if α has the form $\beta+1$, we define $K_\alpha = K_\beta(\tau)\times[0,1]$ where $K_\beta(\tau)$ is the extension of K by $(f_n)_{n\in\tau}$, and τ is picked so that the resulting extension is strong. At limit stages we take inverse limits which also preserve connectedness.

4.1.2 Overview of the chapter

The chapter is organised as follows. We start with section 4.2 containing preliminary definitions and results that will be used throughout the section. Specifically, we include material on disjoint sequences of functions (section 4.2.1), forbidden splittings (section 4.2.2), irreducible maps and Gleason spaces (section 4.2.3, see also section 3.2.2) and extensions of topological spaces (section 4.2.4).

We then proceed to section 4.3 in which we define property (K) and show that a topological space with property (K) is weakly Koszmider. Section 4.4 contains all intermediate results regarding preservation of suprema and forbidden splittings. Finally, section 4.5 is devoted to the construction of a space K which satisfies properties (P1)–(P4), as is checked in section 4.6.

4.1.3 Notation and terminology

- Let K and L be topological spaces and suppose that there exists a natural, unambiguously defined continuous surjection from L onto K. We denote such a surjection by ρ_K^L. For example, if $L = K \times [0,1]$, we define

$$\rho_K^L((x,t)) = x$$

  for each $(x,t) \in L$ with $x \in K$ and $t \in [0,1]$.

- For convenience, whenever α, β are ordinals with $\beta \leqslant \alpha$, we will use the standard notation π_β^α instead of the defined above $\rho_{[0,1]^\beta}^{[0,1]^\alpha}$.

- Suppose that K and L are topological spaces such that there exists a (natural) continuous surjection $\rho_K^L \colon L \twoheadrightarrow K$. For any $f \in \mathscr{C}(K)$ we define the *lifting of f to $\mathscr{C}(L)$* to be the function

$$f^{[L]} = f \circ \rho_K^L,$$

  and for any $\mathcal{F} \subseteq \mathscr{C}(K)$ we set

$$\mathcal{F}^{[L]} = \{f^{[L]} : f \in \mathcal{F}\}.$$

  In a similar fashion, for any $A \subseteq K$ and $\mathcal{A} \subseteq \mathscr{P}(K)$ we define

$$A^{[L]} = (\rho_K^L)^{-1}(A)$$

  to be the *lifting of A to L*, and

$$\mathcal{A}^{[L]} = \{A^{[L]} : A \in \mathcal{A}\}.$$

- If $\mathcal{A} \subseteq \mathscr{P}(K)$ and $I \subseteq [0,1]$, we define

$$\mathcal{A} \times I = \{A \times I : A \in \mathcal{A}\}.$$

4.2 Preliminary results

4.2.1 Bounded disjoint sequences of functions and their suprema

Let K be a topological space and suppose that $(f_n)_{n\in\omega}$ is a disjoint sequence in $\mathscr{C}(K)$ with $\mathrm{range}(f_n) \subseteq [0,1]$ for each n.

Definition 4.2.1. We say that $f \in \mathscr{C}(K)$ is a *supremum of* $(f_n)_{n\in\omega}$ *in* $\mathscr{C}(K)$ if f is the pointwise supremum of the set $\{f_n : n \in \omega\}$. When a supremum of $(f_n)_{n\in\omega}$ exists in $\mathscr{C}(K)$, we denote it by $\vee_{n\in\omega} f_n$.

Lemma 4.2.2. *Suppose that* (f_n) *has a supremum in* $\mathscr{C}(K)$*. Then*

$$\operatorname{supp}(\vee_{n\in\omega} f_n) \subseteq \overline{\bigcup \operatorname{supp}(f_n)}.$$

Proof. Consider any $x \notin \overline{\bigcup \operatorname{supp}(f_n)}$. There exists $g \in \mathscr{C}(K)$ such that

$$[g]|_{\overline{\bigcup \operatorname{supp}(f_n)}} \equiv 1 \quad \text{and} \quad g(x) = 0.$$

Then $f_n \leqslant g$ for each n, and so $0 \leqslant \vee_{n\in\omega} f_n \leqslant g$. In particular, $0 \leqslant \vee_{n\in\omega} f_n(x) \leqslant g(x) = 0$ meaning that $x \notin \operatorname{supp}(\vee_{n\in\omega} f_n)$ as required. □

Koszmider's paper [Kos04] contains a simple criterion regarding how one can check whether a given function $f \in \mathscr{C}(K)$ is a supremum of $(f_n)_{n\in\omega}$. In order to state it, we need to introduce another piece of notation.

Notation 4.2.3. Let $f \in \mathscr{C}(K)$. We define

$$\Delta(f, (f_n)_{n\in\omega}) = \left\{ x \in K : \sum_{n\in\omega} f_n(x) \neq f(x) \right\}.$$

Lemma 4.2.4 ([Kos04, Lemma 4.1 (a)])**.** *Let* $f \in \mathscr{C}(K)$*. Then* f *is a supremum of* $(f_n)_{n\in\omega}$ *in* $\mathscr{C}(K)$ *if and only if* $\Delta(f, (f_n)_{n\in\omega})$ *is nowhere dense in* K*.*

We finish the section with another result from [Kos04].

Notation 4.2.5. Let $\tau \subseteq \omega$. We define

$$\mathcal{D}(\tau) = \bigcup \{U \subseteq K : U \text{ is open and the set } \{n \in \tau : \operatorname{supp}(f_n) \cap U \neq \emptyset\} \text{ is finite}\}.$$

Lemma 4.2.6 ([Kos04, Lemma 4.1 (b)])**.** $\mathcal{D}(\tau)$ *is an open dense subset of* K *and* $\sum_{n\in\tau} f_n$ *is well-defined and continuous on* $\mathcal{D}(\tau)$*.*

4.2.2 Forbidden splittings

Let K be a topological space and $\mathcal{P}, \mathcal{Q} \subseteq \mathscr{P}(K)$.

Definition 4.2.7. We say that

- the pair $(\mathcal{P}, \mathcal{Q})$ is *split in K (by h)* if there exists $h \in \mathscr{C}(K)$ with

$$\bigcup \mathcal{P} \subseteq h^{-1}(\{0\}), \quad \bigcup \mathcal{Q} \subseteq h^{-1}(\{1\}),$$

- the pair $(\mathcal{P}, \mathcal{Q})$ *forms a forbidden splitting in K* if it cannot be split in K.

Using Urysohn's Lemma [Wil70, 15.4], the second part of Definition 4.2.7 can be reformulated as follows.

Lemma 4.2.8. *The pair $(\mathcal{P}, \mathcal{Q})$ forms a forbidden splitting in K if and only if*

$$\overline{\bigcup \mathcal{P}} \cap \overline{\bigcup \mathcal{Q}} \neq \emptyset.$$

4.2.3 Irreducible maps and Gleason spaces revisited

In this section we collate the results about irreducible maps and Gleason spaces which will be used throughout the section. Our notation is carried over from section 3.2.2. For more information we refer the reader to [CN74, Chapter 2].

The first result describes properties of Gleason spaces and, in particular, shows that Gleason spaces can be regarded as a topological analogue of complete Boolean algebras (which is not surprising as a Gleason space is, after all, the Stone space of a complete Boolean algebra).

Theorem 4.2.9. *Let X be a topological space. The following statements are equivalent.*

(i) X is the Gleason space of some topological space Y.

(ii) $\mathscr{C}(X)$ is Dedekind complete, that is, every nonempty bounded subset of $\mathscr{C}(X)$ has a supremum in $\mathscr{C}(X)$.

(iii) X is projective, that is, if Z, Z' are topological spaces, $\rho\colon Z' \twoheadrightarrow Z$ is a continuous surjection and $\psi\colon X \to Z$ is continuous, there exists a continuous $\tilde{\psi}\colon X \to Z'$ such

that the diagram

$$\begin{array}{ccc} & & Z' \\ & \overset{\tilde{\psi}}{\nearrow} & \downarrow \rho \\ X & \xrightarrow[\psi]{} & Z \end{array}$$

is commutative, that is, $\psi = \rho\tilde{\psi}$*.*

(iv) X *is extremally disconnected.*

Proof. A proof of equivalence of (ii), (iii) and (iv) may be found in [Sem71, Theorem 24.7.1], and it was shown in [Gle58] that (i) and (iv) imply each other. ☐

Corollary 4.2.10. *Let* X*,* Z *and* Z' *be topological spaces such that there exist continuous irreducible surjections* $\psi\colon \mathscr{G}_X \twoheadrightarrow Z$ *and* $\rho\colon Z' \twoheadrightarrow Z$*. Then there exists a continuous irreducible surjection* $\tilde{\psi}\colon \mathscr{G}_X \twoheadrightarrow Z'$ *which makes the diagram*

$$\begin{array}{ccc} & & Z' \\ & \overset{\tilde{\psi}}{\nearrow} & \downarrow \rho \\ \mathscr{G}_X & \xrightarrow[\psi]{} & Z \end{array}$$

commutative.

Proof. Existence of a continuous $\tilde{\psi}$ with $\psi = \rho\tilde{\psi}$ follows from Theorem 4.2.9. If $\tilde{\psi}(\mathscr{G}_X) \subsetneqq Z'$ then, by irreducibility of ρ, we have

$$Z = \psi(\mathscr{G}_X) = \rho(\tilde{\psi}(\mathscr{G}_X)) \subsetneqq Z$$

which is, of course, a contradiction. Thus $\tilde{\psi}$ is surjective.

Suppose now that $F \subsetneqq \mathscr{G}_X$ is closed. Irreducibility of ψ implies that

$$\rho(\tilde{\psi}(F)) = \psi(F) \subsetneqq Z = \rho(\tilde{\psi}(\mathscr{G}_X))$$

which means that $\tilde{\psi}(F) \subsetneqq \tilde{\psi}(\mathscr{G}_X)$ and $\tilde{\psi}$ is irreducible. ☐

We finish the section with two easy but useful results.

Lemma 4.2.11. *For* $i \in \{0, 1\}$ *let* $\rho_i\colon X_i \twoheadrightarrow Y_i$ *be a continuous irreducible surjection. Define*

$$\begin{array}{rcl} \rho\colon X_0 \times X_1 & \to & Y_0 \times Y_1 \\ (x_0, x_1) & \mapsto & (\rho_0(x_0), \rho_1(x_1)). \end{array}$$

Then ρ is a continuous irreducible surjection.

Proof. Firstly, ρ is continuous and onto as both ρ_0 and ρ_1 are. To prove irreducibility, take any closed $F \subsetneq X_0 \times X_1$. Then for each i there exists open nonempty $U_i \subseteq X_i$ such that

$$U_0 \times U_1 \subseteq (X_0 \times X_1) \backslash F.$$

The set $X_i \backslash U_i$ is a proper closed subset of X_i. Thus, by irreducibility of ρ_i, we can pick $y_i \in Y_i \backslash \rho_i(X_i \backslash U_i)$. Consider now the point $y = (y_0, y_1)$ and suppose that $x = (x_0, x_1) \in \rho^{-1}(\{y\})$. Then for each i we have $\rho_i(x_i) = y_i$ and so $x_i \in U_i$. Consequently $x \in U_0 \times U_1$, $y \notin \rho(F)$ and so $\rho(F) \subsetneq X_0 \times X_1$. $\square$

Lemma 4.2.12. *Let $\rho\colon X \to Y$ be a continuous irreducible surjection.*

(i) If D is an open dense subset of Y, then $\rho^{-1}(D)$ is an open dense subset of X.

(ii) If M is a nowhere dense subset of Y, then $\rho^{-1}(M)$ is a nowhere dense subset of X.

Proof. (i) First of all, continuity of ρ implies that $\rho^{-1}(D)$ is open. Now, consider any open U with $\emptyset \neq U \subseteq X$. Then $X \backslash U$ is a proper closed subset of X and so, by irreducibility, $\rho(X \backslash U)$ is a proper closed subset of Y. Thus we can find a point

$$d \in D \cap (Y \backslash \rho(X \backslash U)).$$

Then $\rho^{-1}(\{d\}) \subseteq U$ and $U \cap \rho^{-1}(D) \neq \emptyset$. Consequently, $\rho^{-1}(D)$ is dense in X.

(ii) If $\operatorname{int}\left(\overline{M}\right) = \emptyset$, then

$$Y = Y \backslash \operatorname{int}\left(\overline{M}\right) = \overline{Y \backslash \overline{M}}$$

which means that $Y \backslash \overline{M}$ is an open dense subset of Y. By the previous part, $\rho^{-1}(Y \backslash \overline{M})$ is an open dense subset of X, that is,

$$X = \overline{\rho^{-1}(Y \backslash \overline{M})} = \overline{X \backslash \rho^{-1}(\overline{M})} = X \backslash \operatorname{int}\left(\rho^{-1}(\overline{M})\right)$$

and so, in particular,

$$\operatorname{int}\left(\overline{\rho^{-1}(M)}\right) \subseteq \operatorname{int}\left(\rho^{-1}(\overline{M})\right) = \emptyset.$$

$\square$

4.2.4 Extensions of topological spaces

Let K be a topological space and suppose that $(f_n)_{n\in\omega}$ is a disjoint sequence in $\mathscr{C}(K)$ with range$(f_n) \subseteq [0,1]$ for each n.

Definition 4.2.13 ([Kos04, Definition 4.2])**.** Let $\tau \subseteq \omega$.

- We define *the extension* $K(\tau)$ *of* K *by* $(f_n)_{n\in\tau}$ to be the set
$$K(\tau) = \overline{\operatorname{graph}\left(\left[\sum_{n\in\tau} f_n\right]\Bigg|_{\mathcal{D}(\tau)}\right)},$$
where the closure is taken in $K \times [0,1]$.
- We say that the extension $K(\tau)$ of K by $(f_n)_{n\in\tau}$ is *strong* if
$$\operatorname{graph}\left(\sum_{n\in\tau} f_n\right) \subseteq K(\tau).$$

Notation 4.2.14. Note that density of $\mathcal{D}(\tau)$ ensures that the projection $(x,t) \mapsto x$ from $K(\tau)$ into K is onto. Following the notation from section 4.1.3, we denote it by $\rho_K^{K(\tau)}$.

Theorem 4.2.15 ([Kos04, Lemma 4.3 (a)])**.** *The (continuous) function*

$$\begin{aligned} f\colon K(\tau) &\to [0,1] \\ (x,t) &\mapsto t \qquad \forall (x,t) \in K(\tau) \text{ with } x \in K, t \in [0,1] \end{aligned}$$

is a supremum of $(f_n^{[K(\tau)]})_{n\in\tau}$ *in* $\mathscr{C}(K(\tau))$.

Notation 4.2.16. We denote the function f from Theorem 4.2.15 by $f_\tau^{[K(\tau)]}$, that is,

$$f_\tau^{[K(\tau)]}(x,t) = t \quad \forall (x,t) \in K(\tau). \tag{4.1}$$

The extension $K(\tau)$ turns out to be a natural analogue of the algebra $\langle \mathfrak{A}, A_\tau \rangle$.

Proposition 4.2.17. *Let* $K(\tau)$ *be the extension of* K *by* $(f_n)_{n\in\tau}$*. Then*

$$\mathscr{C}(K(\tau)) = \overline{\left\langle \mathscr{C}(K)^{[K(\tau)]}, f_\tau^{[K(\tau)]} \right\rangle}.$$

Proof. Let (x,t), (x',t') be distinct points in $K(\tau)$. Then either $x \neq x'$ and there exists $g \in \mathscr{C}(K)$ with $g(x) \neq g(x')$ giving

$$g^{[K(\tau)]}((x,t)) = g \circ \rho_K^{K(\tau)}((x,t)) = g(x) \neq g(x') = g \circ \rho_K^{K(\tau)}((x',t')) = g^{[K(\tau)]}(x',t'),$$

or $t \neq t'$ and

$$f_\tau^{[K(\tau)]}((x,t)) = t \neq t' = f_\tau^{[K(\tau)]}((x',t')).$$

Thus $\left\langle \mathscr{C}(K)^{[K(\tau)]}, f_\tau^{[K(\tau)]} \right\rangle$ separates points of $K(\tau)$. Since $\chi_{\mathscr{C}(K(\tau))} = \chi_{\mathscr{C}(K)}^{[K(\tau)]}$, the result follows from the Stone–Weierstrass Theorem [DS58, IV.6.15, Theorem 16]. □

Corollary 4.2.18. *Let D be a dense subset of $\mathscr{C}(K)$. Then*

$$\mathscr{C}(K(\tau)) = \overline{\left\langle D^{[K(\tau)]}, f_\tau^{[K(\tau)]} \right\rangle}.$$

In particular,

$$\text{density}(\mathscr{C}(K(\tau))) = \text{density}(\mathscr{C}(K)).$$

Proof. Let $f \in \mathscr{C}(K(\tau))$ and $\varepsilon > 0$. By Proposition 4.2.17, we can find $k \in \omega$ and $g_0, \dots, g_k \in \mathscr{C}(K)$ such that if we define

$$g = g_0^{[K(\tau)]} + g_1^{[K(\tau)]} f_\tau^{[K(\tau)]} + \dots + g_k^{[K(\tau)]} \left(f_\tau^{[K(\tau)]} \right)^k,$$

then

$$\|f - g\|_\infty \leqslant \frac{\varepsilon}{2}.$$

By density, for each $j \leqslant k$ we may find $h_j \in D$ with

$$\|g_j - h_j\| \leqslant \frac{\varepsilon}{2^{j+2}}.$$

Then, defining

$$h = h_0^{[K(\tau)]} + h_1^{[K(\tau)]} f_\tau^{[K(\tau)]} + \dots + h_k^{[K(\tau)]} \left(f_\tau^{[K(\tau)]} \right)^k \in \left\langle D^{[K(\tau)]}, f_\tau^{[K(\tau)]} \right\rangle$$

and keeping in mind that $\left\| f_\tau^{[K(\tau)]} \right\|_\infty \leqslant 1$, we get

$$\begin{aligned}
\|f - h\|_\infty &\leqslant \|f - g\|_\infty + \|g - h\|_\infty \\
&\leqslant \|f - g\|_\infty + \|g_0 - h_0\|_\infty + \|g_1 - h_1\|_\infty \left\| f_\tau^{[K(\tau)]} \right\|_\infty + \cdots \\
&\quad + \|g_k - h_k\|_\infty \left(\left\| f_\tau^{[K(\tau)]} \right\|_\infty \right)^k \\
&< \frac{\varepsilon}{2} + \sum_{j=0}^{\infty} \frac{\varepsilon}{2^{j+2}} \\
&= \varepsilon,
\end{aligned}$$

which completes the proof. □

The main reason we are interested in strong extensions rather than general ones is that the former preserve connectedness and so will help us to achieve (P4).

Theorem 4.2.19 ([Kos04, Lemma 4.4])**.** *Suppose that the extension $K(\tau)$ is strong. If K is connected, then so is $K(\tau)$.*

Strong extensions give us another bonus which will be used later in the section on preservation of forbidden splittings.

Lemma 4.2.20. *Suppose that the extension $K(\tau)$ is strong and let $(x,t) \in K(\tau)$ be such that $x \in \mathcal{D}(\tau)$. Then $t = \sum_{n\in\tau} f_n(x)$.*

Proof. The proof is essentially the same as the proof of part (a) of [Kos04, Lemma 4.3].

Suppose, for a contradiction, that $t \neq \sum_{n\in\tau} f_n(x)$. Let $U \ni t$ and $V \ni \sum_{n\in\tau} f_n(x)$ be disjoint open sets. By Lemma 4.2.6, the function $\sum_{n\in\tau} f_n$ is continuous at x, and so there exists an open $W \ni x$ with

$$\sum_{n\in\tau} f_n(W) \subseteq V.$$

Consider now the set

$$A = W \times U.$$

Then A is open and $A \cap K(\tau) \ni (x,t)$. Hence we can find a point

$$(x',t') \in A \cap \operatorname{graph}\left(\left[\sum_{n\in\tau} f_n\right]\Big|_{\mathcal{D}(\tau)}\right).$$

Then $x' \in W$ and thus

$$U \ni t' = \sum_{n\in\tau} f_n(x') \in \sum_{n\in\tau} f_n(W) \subseteq V.$$

This, however, lead to a contradiction as we assumed that $U \cap V = \emptyset$. □

Corollary 4.2.21. *Suppose that the extension $K(\tau)$ is strong. Then $\rho_K^{K(\tau)}$ is a continuous irreducible surjection.*

Proof. Surjectivity of $\rho_\tau^{K(\tau)}$ was already mentioned in Notation 4.2.14. Consider now any closed $F \subsetneqq K(\tau)$. Then $K(\tau)\backslash F$ is nonempty and open and hence there exists a point

$$(x,t) \in (K(\tau)\backslash F) \cap \operatorname{graph}\left(\left[\sum_{n\in\tau} f_n\right]\Bigg|_{\mathcal{D}(\tau)}\right).$$

Then $x \in \mathcal{D}(\tau)$ and, by Lemma 4.2.20, $\left(\rho_K^{K(\tau)}\right)^{-1}(\{x\}) = \{(x,t)\}$. Thus $x \in K\backslash\rho_K^{K(\tau)}(F)$ and $\rho_K^{K(\tau)}(F) \subsetneqq K$ as required. □

Note that Corollary 4.2.21 together with Lemma 4.2.12 implies that if the extension $K(\tau)$ is strong, then $\left(\rho_K^{K(\tau)}\right)^{-1}(M)$ is nowhere dense in $K(\tau)$ whenever M is nowhere dense in K. This result, in fact, holds for an arbitrary extension.

Lemma 4.2.22 ([Kos04, Lemma 4.3 (a)]). *Let M be a nowhere dense subset of K. Then $\left(\rho_K^{K(\tau)}\right)^{-1}(M)$ is a nowhere dense subset of $K(\tau)$.*

Theorem 4.2.23 ([Kos04, Lemma 4.5 (a)]). *Fix an almost disjoint family $(\sigma_\xi)_{\xi<2^\omega}$ of infinite subsets of ω and assume that* $\operatorname{weight}(K) = \kappa < 2^\omega$. *There exists a set $\mathcal{A} \subseteq 2^\omega$ with $|\mathcal{A}| \leqslant \kappa$ such that for any $\xi \in 2^\omega\backslash\mathcal{A}$ and any infinite $\tau \subseteq \sigma_\xi$ the extension $K(\tau)$ is strong.*

4.3 Property (K)

Property (K) is a natural generalisation of property (K′) introduced in section 3.3.

Definition 4.3.1. Let K be a topological space. We say that K *has property (K)* if, given

(a) a disjoint sequence (f_n) in $\mathscr{C}(K)$ with $\operatorname{range}(f_n) \subseteq [0,1]$ for each n,

(b) a disjoint sequence (U_n) of open nonempty subsets of K with $\operatorname{supp}(f_m) \cap U_n = \emptyset$ for all m, n,

there exists an infinite $\tau \subseteq \omega$ such that

(i) the sequence $(f_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$,

(ii) the pair $((U_n)_{n\in\tau}, (U_n)_{n\notin\tau})$ cannot be split in K.

Theorem 4.3.2. *Let K be a topological space with property (K). Then K is weakly Koszmider.*

The proof of Theorem 4.3.2 is similar to the proof of Theorem 3.3.2 and so, as before, we start with an assumption which abuses the existing notation slightly but nevertheless allows us to write out arguments in a more simple and aesthetically pleasing manner.

Assumption 4.3.3 (Valid until the rest of section 4.3)**.** Throughout the rest of this section

(i) unless stated otherwise, all cited subsets of ω (such as τ, σ and similar) are assumed to be infinite,

(ii) unless stated otherwise, all cited subsets of K (such as U_n, X and similar) are assumed to be nonempty and open,

(iii) if (U_n) is a sequence in K, we say that $(V_n) \subseteq K$ is a *refinement* of (U_n) if $V_n \subseteq U_n$ for each n. When this is the case, we write $(V_n) \preccurlyeq (U_n)$. As discussed above, it is implicitly assumed that all U_n and V_n are nonempty and open.

Lemma 4.3.4. *Let K be a topological space and suppose that we are given*

(a) an operator $T \colon \mathscr{C}(K) \to \mathscr{C}(K)$,

(b) a disjoint sequence (f_n) in $\mathscr{C}(K)$ with $\operatorname{range}(f_n) \subseteq [0, 1]$ *for each n,*

(c) a sequence (U_n) of mutually disjoint (nonempty open) subsets of K,

(d) $\varepsilon > 0$.

Then there exist

(i) (an infinite) $\sigma \subseteq \omega$,

(ii) a refinement $(D_n)_{n\in\sigma} \preccurlyeq (U_n)_{n\in\sigma}$

with the property that whenever τ is a subset of σ such that $(f_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$ and $n \in \sigma\backslash\tau$, we have

$$\left\| [T(\vee_{n\in\tau} f_n)]|_{D_n} \right\|_\infty \leqslant \varepsilon.$$

The proof of Lemma 4.3.4 requires a few more intermediate results which we write out as separate claims.

Claim 4.3.5. *Let $(g_n)_{n\in\omega} \subseteq \mathscr{C}(K)$ be a disjoint sequence with* $\mathrm{range}(g_n) \subseteq [0,1]$ *for each n, and consider any $\tau, \theta \subseteq \omega$ with the property that $(g_n)_{n\in\tau}$ and $(g_n)_{n\in\theta}$ have suprema in $\mathscr{C}(K)$. Then*

(i) if $\tau \cap \theta = \emptyset$, then $(\vee_{n\in\tau} g_n) \cdot (\vee_{n\in\theta} g_n) = 0$.

Suppose now that $F \subseteq \tau$ is finite. Then

(ii) $(g_n)_{n\in F}$ has a supremum in $\mathscr{C}(K)$ and $\vee_{n\in F} g_n = \sum_{n\in F} g_n$,

(iii) $(g_n)_{n\in\tau\setminus F}$ has a supremum in $\mathscr{C}(K)$ and $\vee_{n\in\tau} g_n = \vee_{n\in\tau\setminus F} g_n + \vee_{n\in F} g_n$.

Proof. (i) By Lemma 4.2.2, $\mathrm{supp}(\vee_{n\in\tau} g_n) \cap \mathrm{supp}(\vee_{n\in\theta} g_n)$ is an open set contained in $\overline{\bigcup_{n\in\tau} \mathrm{supp}(g_n)} \cap \overline{\bigcup_{n\in\theta} \mathrm{supp}(g_n)}$ and so the result follows from disjointness of (g_n).

(ii) This part is immediate as $\sum_{n\in F} g_n \in \mathscr{C}(K)$.

(iii) Consider the function

$$h = \vee_{n\in\tau} g_n - \vee_{n\in F} g_n = \vee_{n\in\tau} g_n - \sum_{n\in F} g_n.$$

Since $\vee_{n\in\tau} g_n$ and $\vee_{n\in F} g_n$ both lie in $\mathscr{C}(K)$, so does h. We also have

$$\begin{aligned}
\Delta(h, (g_n)_{n\in\tau\setminus F}) &= \left\{ x \in K : h(x) \neq \sum_{n\in\tau\setminus F} g_n(x) \right\} \\
&= \left\{ x \in K : \vee_{n\in\tau} g_n(x) - \sum_{n\in F} g_n(x) \neq \sum_{n\in\tau\setminus F} g_n(x) \right\} \\
&= \left\{ x \in K : \vee_{n\in\tau} g_n(x) \neq \sum_{n\in\tau} g_n(x) \right\} \\
&= \Delta(\vee_{n\in\tau} g_n, (g_n)_{n\in\tau}).
\end{aligned}$$

By Lemma 4.2.4, $\Delta(\vee_{n\in\tau} g_n, (g_n)_{n\in\tau})$ is nowhere dense, hence so is $\Delta(h, (g_n)_{n\in\tau\setminus F})$. Thus, by Lemma 4.2.4 again, h is a supremum of $(f_n)_{n\in\tau\setminus F}$ in $\mathscr{C}(K)$. □

Claim 4.3.6. *Suppose that we are given an operator $S\colon \mathscr{C}(K) \to \mathscr{C}(K)$, a disjoint $(g_n) \subseteq \mathscr{C}(K)$ with* $\mathrm{range}(g_n) \subseteq [0,1]$ *for each n, a disjoint $(Y_n) \subseteq \mathscr{P}(K)$ and $\delta > 0$. Then*

(i) there exist $M \in \omega$ and a refinement $(Y_n^{(1)}) \preccurlyeq (Y_n)$ such that

$$\left\| [Sg_M]|_{Y_n^{(1)}} \right\|_\infty \leqslant \delta \quad \textit{for infinitely many } n,$$

(ii) for any $N \in \omega$ there exist $\sigma_N \subseteq \omega$ and $Y_N^{(2)} \subseteq Y_N$ such that whenever τ is a subset of σ_N with the property that $(g_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$, we have

$$\left\| [S(\vee_{n\in\tau} g_n)]|_{Y_N^{(2)}} \right\|_\infty \leqslant \delta.$$

Proof. (i) Suppose, for a contradiction, that for any $M \in \omega$ and $(Y_n^{(1)}) \preccurlyeq (Y_n)$ there exists k such that

$$\left\| [Sg_M]|_{Y_n^{(1)}} \right\|_\infty > \delta \quad \forall n \geqslant k.$$

By Claim 3.3.5, for each $n \geqslant k$ there exist $\alpha \in \{-1, 1\}$ and $\hat{Y}_n^{(1)} \subseteq Y_n^{(1)}$ with the property

$$[S(\alpha g_M)]|_{\hat{Y}_n^{(1)}} > \delta. \tag{4.2}$$

To get a contradiction, we construct inductively sequences $(k_j)_{j\in\omega}$, $((\alpha_{j,n})_{n\geqslant k_j})_{j\in\omega}$ and $((Y_{j,n})_{n\geqslant k_j})_{j\in\omega}$ such that for each j we have

(i.a) $k_{j+1} > k_j$,

(i.b) $(\alpha_{j,n})_{n\geqslant k_j} \subseteq \{-1, 1\}$,

(i.c) $(Y_{j+1,n})_{n\geqslant k_{j+1}} \preccurlyeq (Y_{j,n})_{n\geqslant k_{j+1}} \preccurlyeq (Y_n)_{n\geqslant k_{j+1}}$,

(i.d) $[S(\alpha_{j,n} g_j)]|_{Y_{j,n}} > \delta$ for all $n \geqslant k_j$.

Base Case. By (4.2), there exists $k_0 \in \omega$ such that for each $n \geqslant k_0$ we can find $\alpha_{0,n} \in \{-1, 1\}$ and $Y_{0,n} \subseteq Y_n$ with

$$[S(\alpha_{0,n} g_0)]|_{Y_{0,n}} > \delta.$$

Inductive Step. Suppose that the construction has been completed up to some j. Applying (4.2) to $j+1$ and $(Y_{j,n})_{n\geqslant k_j}$, we can pick $k \in \omega$ such that for each $n \geqslant k$ there exist $\alpha_{j+1,n} \in \{-1, 1\}$ and $Y_{j+1,n} \subseteq Y_{j,n}$ with

$$[S(\alpha_{j+1,n} g_{j+1})]|_{Y_{j+1,n}} > \delta.$$

To complete the inductive step, take $k_{j+1} = \max\{k_j + 1, k\}$ to ensure $k_{j+1} > k_j$.

Once the construction has been completed, take $J = \left\lceil \frac{\|S\|}{\delta} \right\rceil$. It follows from disjointness of (g_n) that for any combination of signs the function $\pm g_0 \pm \cdots \pm g_J$ has norm at most 1 and, in particular,

$$\|S(\alpha_{0,k_J} g_0 + \cdots + \alpha_{J,k_J} g_J)\|_\infty \leqslant \|S\|.$$

On the other hand, since $(k_j)_{0\leqslant j\leqslant J}$ is an increasing sequence, while $(Y_{j,k_J})_{0\leqslant j\leqslant J}$ is decreasing, condition (i.d) implies that

$$[S(\alpha_{0,k_J}g_0+\cdots+\alpha_{J,k_J}g_J)]|_{Y_{J,k_J}}>(J+1)\delta>\frac{\|S\|}{\delta}\delta=\|S\|\,,$$

which, combined with the previous estimate, gives the required contradiction.

(ii) Fix $N\in\omega$ and suppose, for a contradiction, that for any $\sigma\subseteq\omega$ and $Y_N^{(2)}\subseteq Y_N$ there exists $\tau\subseteq\sigma$ such that $(g_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$ and

$$\left\|\,[S(\vee_{n\in\tau}g_n)]|_{Y_N^{(2)}}\right\|_\infty>\delta.$$

Then we can find $\alpha\in\{-1,1\}$ and $\hat{Y}_N^{(2)}\subseteq Y_N^{(2)}$ such that

$$[S\,(\alpha\vee_{n\in\tau}g_n)]|_{\hat{Y}_N^{(2)}}>\delta. \tag{4.3}$$

Define $J=\left[\frac{\|S\|}{\delta}\right]$ and partition ω into $J+1$ disjoint parts $\sigma_0,\ldots,\sigma_J$. Using another inductive argument, we construct sequences $(\tau_j)_{0\leqslant j\leqslant J}$, $(\alpha_j)_{0\leqslant j\leqslant J}$ and $(Y_{j,N})_{0\leqslant j\leqslant J}$ such that for each j

(ii.a) $\tau_j\subseteq\sigma_j$,

(ii.b) $\alpha_j\in\{-1,1\}$,

(ii.c) $Y_{j+1,N}\subseteq Y_{j,N}\subseteq Y_N$,

(ii.d) $(g_n)_{n\in\tau_j}$ has a supremum in $\mathscr{C}(K)$ and $\left[S\left(\alpha_j\vee_{n\in\tau_j}g_n\right)\right]\big|_{Y_{j,N}}>\delta$.

The *Base Case* follows from (4.3) applied to σ_0 and Y_N. For the *Inductive Step* suppose that the construction has been completed up to some j with $0\leqslant j<J$ and apply (4.3) to σ_{j+1} and $Y_{j,N}$.

We now proceed as in part (i). Disjointness of $(\sigma_j)_{0\leqslant j\leqslant J}$ and part (i) of Claim 4.3.5 imply that $(\vee_{n\in\tau_j}g_n)_{0\leqslant j\leqslant J}$ is a disjoint sequence, and hence

$$\|S\,(\alpha_0\vee_{n\in\tau_0}g_n+\cdots+\alpha_J\vee_{n\in\tau_J}g_n)\|_\infty\leqslant\|S\|\,\|\alpha_0\vee_{n\in\tau_0}g_n+\cdots+\alpha_J\vee_{n\in\tau_J}g_n\|\leqslant\|S\|\,,$$

while, on the other hand, since $(Y_{j,N})_{0\leqslant j\leqslant J}$ is decreasing, condition (ii.d) implies that

$$[S\,(\alpha_0\vee_{n\in\tau_0}g_n+\cdots+\alpha_J\vee_{n\in\tau_J}g_n)]|_{Y_{J,N}}>(J+1)\delta>\frac{\|S\|}{\delta}\delta=\|S\|$$

giving the required contradiction. ∎

Proof of Lemma 4.3.4. The proof is constructive and we start with showing that there exist a subsequence (n_j) and a refinement $(C_{n_j}) \preccurlyeq (U_{n_j})$ with the property that

$$\left\| [Tf_{n_k}]|_{C_{n_l}} \right\|_\infty \leqslant \frac{\varepsilon}{2^{k+2}} \quad \forall k, l : k < l. \tag{4.4}$$

To do this, we construct inductively sequences $(n_j)_{j\in\omega}$, $(\Lambda_j)_{j\in\omega}$ and $((U_{j,n})_{n\in\Lambda_j})_{j\in\omega}$ such that for each j

(i.a) $n_{j+1} > n_j$,

(i.b) $\Lambda_{j+1} \subseteq \Lambda_j \subseteq \omega$,

(i.c) $\Lambda j \ni n_{j+1}$,

(i.d) $n_j < \min \Lambda_j$,

(i.e) $(U_{j+1,n})_{n\in\Lambda_{j+1}} \preccurlyeq (U_{j,n})_{n\in\Lambda_{j+1}} \preccurlyeq (U_n)_{n\in\Lambda_{j+1}}$,

(i.f) $\left\| [Tf_{n_j}]|_{U_{j,n}} \right\|_\infty \leqslant \dfrac{\varepsilon}{2^{j+2}}$ for all $n \in \Lambda_j$.

Base Case. By Claim 4.3.6 (i), there exist $n_0 \in \omega$ and $(U_{0,n}) \preccurlyeq (U_n)$ such that

$$\left\| [Tf_{n_0}]|_{U_{0,n}} \right\|_\infty \leqslant \frac{\varepsilon}{4} \quad \text{for infinitely many } n. \tag{4.5}$$

We also define

$$\Lambda_0 = \{n > n_0 : (4.5) \text{ holds for } n\}.$$

Inductive Step. Suppose that the construction has been carried out up to some j. Applying part (i) of Claim 4.3.6 again, we can find $n_{j+1} \in \Lambda_j$ (in particular, $n_{j+1} > n_j$) and $(U_{j+1,n})_{n\in\Lambda_j} \preccurlyeq (U_{j,n})_{n\in\Lambda_j}$ such that

$$\left\| [Tf_{n_{j+1}}]|_{U_{j+1,n}} \right\|_\infty \leqslant \frac{\varepsilon}{2^{j+3}} \quad \text{for infinitely many } n \in \Lambda_j. \tag{4.6}$$

To complete the inductive step, we define

$$\Lambda_{j+1} = \{n \in \Lambda_j : n > n_j \text{ and } (4.6) \text{ holds for } n\}.$$

Once the construction is completed, for each j we define

$$C_{n_j} = U_{j,n_j}.$$

Note that if we pick any k, l with $k < l$, then $n_l \in \Lambda_{l-1} \subseteq \Lambda_k$ and so

$$\left\| [Tf_{n_k}]|_{C_{n_l}} \right\|_\infty = \left\| [Tf_{n_k}]|_{U_{l,n_l}} \right\|_\infty \leqslant \left\| [Tf_{n_k}]|_{U_{k,n_l}} \right\|_\infty \leqslant \frac{\varepsilon}{2^{k+2}}$$

giving (4.4).

We now need to further reduce the size of (C_{n_j}) and for this we construct inductively a subsequence (n_{j_r}) of (n_j) and sequences (σ_r) and $(D_{n_{j_r}})$ such that for each r

(ii.a) $n_{j_{r+1}} > n_{j_r}$,

(ii.b) $\sigma_{r+1} \subseteq \sigma_r \subseteq (n_j)$,

(ii.c) $n_{j_{r+1}} \in \sigma_r$,

(ii.d) $D_{n_{j_r}} \subseteq C_{n_{j_r}}$,

(ii.e) whenever $\tau \subseteq \sigma_r$ is such that $(f_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$, we have

$$\left\| [T(\vee_{n\in\tau} f_n)]|_{D_{n_{j_r}}} \right\|_\infty \leqslant \frac{\varepsilon}{2}. \tag{4.7}$$

For the *Base Case* set $n_{j_0} = n_0$ and apply Claim 4.3.6 (ii) to n_{j_0}, $(f_n)_{n\in(n_j)}$, $(C_n)_{n\in(n_j)}$ and $\varepsilon/2$. For the *Inductive Step* set $n_{j_{r+1}}$ to be the $(r+2)^{\text{nd}}$ element of σ_r (that is, if $\sigma_r = (\sigma_{r,k})_{k\in\omega}$ with $\sigma_{r,k+1} > \sigma_{r,k}$ for each k, then $n_{j_{r+1}} = \sigma_{r,r+1}$) and apply the same result to $n_{j_{r+1}}$, $(f_n)_{n\in\sigma_r}$, $(C_n)_{n\in\sigma_r}$ and $\varepsilon/2$.

Once the construction has been completed, we define

$$\sigma = (n_{j_r})_{r\in\omega}.$$

Suppose now that $\tau \subseteq \sigma$ is such that $(f_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$, and $n \in \sigma\backslash\tau$. By construction, there exists $l \in \omega$ with

$$n = n_{j_l}.$$

Define

$$\tau' = \tau \cap (n_{j_r})_{r>l}.$$

Then $\tau' \subseteq \sigma_l$ and $\tau\backslash\tau' \subseteq (n_{j_r})_{r<l}$. By Claim 4.3.5, both $(f_m)_{m\in\tau'}$ and $(f_m)_{m\in\tau\backslash\tau'}$ have suprema in $\mathscr{C}(K)$ and

$$\vee_{m\in\tau} f_m = \vee_{n\in\tau'} f_m + \sum_{m\in\tau\backslash\tau'} f_m.$$

Putting together all of the above, we get

$$\begin{aligned}
\left\| [T(\vee_{m\in\tau} f_m)]|_{D_n} \right\|_\infty &= \left\| \left[T\left(\vee_{m\in\tau'} f_m + \sum_{m\in\tau\setminus\tau'} f_m \right) \right] \Big|_{D_n} \right\|_\infty \\
&\leqslant \left\| [T(\vee_{m\in\tau'} f_m)]|_{D_{n_{j_l}}} \right\|_\infty + \sum_{\substack{n_{j_k}\in\tau\setminus\tau' \\ k<l}} \left\| \left[T f_{n_{j_k}} \right] \Big|_{C_{n_{j_l}}} \right\|_\infty \\
&\leqslant \underbrace{\frac{\varepsilon}{2}}_{\text{by (4.7)}} + \underbrace{\sum_{\substack{n_{j_k}\in\tau\setminus\tau' \\ k<l}} \frac{\varepsilon}{2^{j_k+1}}}_{\text{by (4.4), as } k<l \Leftrightarrow j_k<j_l} \\
&\leqslant \frac{\varepsilon}{2} + \sum_{\substack{n_{j_k}\in\tau\setminus\tau' \\ k<l}} \frac{\varepsilon}{2^{k+2}} \\
&< \frac{\varepsilon}{2} + \sum_{k\in\omega} \frac{\varepsilon}{2^{k+2}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

as required. □

Finally, as in the zero-dimensional case, we need one more result.

Lemma 4.3.7. *Let K be a topological space and suppose that an operator $T\colon \mathscr{C}(K) \to \mathscr{C}(K)$ is noncentripetal. Then*

(i) there exist $(x_n) \subseteq K$, a disjoint $(f_n) \subseteq \mathscr{C}(K)$ and an $\varepsilon > 0$ such that

$$x_n \notin \overline{\operatorname{supp}(f_n)}, \quad \operatorname{range}(f_n) \subseteq [0,1] \quad \textit{and} \quad |(Tf_n)(x_n)| > \varepsilon \quad \forall n,$$

(ii) there exist a disjoint sequence of nonempty $(U_n) \subseteq \mathscr{P}(K)$, a disjoint $(f_n) \subseteq \mathscr{C}(K)$ and $\varepsilon > 0$ such that

$$\operatorname{range}(f_n) \subseteq [0,1], \quad \operatorname{supp}(f_m) \cap U_n = \emptyset \quad \textit{and} \quad \left| [Tf_n]|_{U_n} \right| > \varepsilon \quad \forall m, n.$$

Proof. (i) By part (ii) of Lemma 2.3.2, there exist $(x_n) \subseteq K$, a disjoint $(f_n) \subseteq \mathscr{C}(K)$, $M > 0$ and $\varepsilon > 0$ such that

$$x_n \notin \overline{\operatorname{supp}(f_n)}, \quad \|f_n\|_\infty \leqslant M \quad \text{and} \quad |(Tf_n)(x_n)| > 2\varepsilon \quad \forall n.$$

Replacing f_n with f_n/M, we may assume that $\|f_n\|_\infty \leqslant 1$. Thus range$(f_n^+)$, range$(f_n^-) \subseteq [0,1]$. Consequently, either $|(Tf_n^+)(x_n)| > \varepsilon$ or $|(Tf_n^-)(x_n)| > \varepsilon$ (or both). Thus, replacing f_n with f_n^+ or f_n^- and passing to subsequences if needed, we may assume that range$(f_n) \subseteq [0,1]$ for each n.

(ii) Let (x_n), (f_n) and ε be as in part (i). By disjointness, the norm of any finite sum of $\pm f_n$ is at most 1. Hence, by boundedness of T, for any n there exist only finitely many m with $x_n = x_m$. Thus, without loss of generality, we may assume that all x_n are distinct.

Pick any $n \in \omega$. By continuity, we can find V_n with $x_n \in V_n \subseteq K \backslash \mathrm{supp}(f_n)$ and

$$\left| [Tf_n]|_{V_n} \right| > \varepsilon.$$

If $\mathrm{supp}(f_m) \cap V_n = \emptyset$ for all m, n, we set $U_n = V_n$ for each n. Otherwise there are two possibilities.

Case 1. There exist $m \in \omega$ and a subsequence (n_j) such that

$$\mathrm{supp}(f_m) \cap V_{n_j} \neq \emptyset \quad \forall j.$$

In this case for each j we define

$$U_{n_j} = \mathrm{supp}(f_m) \cap V_{n_j}.$$

Then (U_{n_j}) is disjoint and

$$\left| [Tf_{n_j}]|_{U_{n_j}} \right| > \varepsilon \quad \forall j.$$

Furthermore, since $\mathrm{supp}(f_{n_j}) \cap V_{n_j} = \emptyset$, we have $m \notin \{n_j : j \in \omega\}$. Consequently,

$$\bigcup \mathrm{supp}(f_{n_j}) \cap \bigcup U_{n_j} \subseteq \bigcup \mathrm{supp}(f_{n_j}) \cap \mathrm{supp}(f_m) = \emptyset,$$

and so (f_{n_j}), (U_{n_j}) provide the required pair.

Case 2. For each m there exists N_m such that

$$\mathrm{supp}(f_m) \cap V_n = \emptyset \quad \forall n \geqslant N_m.$$

In this case we start with constructing inductively a subsequence (n_j) such that

$$\mathrm{supp}(f_{n_k}) \cap V_{n_l} = \emptyset \quad \forall j, k, l : k \leqslant l \leqslant j. \tag{4.8}$$

For this we define $n_0 = 0$ and, assuming that $n_0 < n_1 < \cdots < n_j$ have been constructed, set $n_{j+1} = \max\{N_0, \ldots, N_{n_j}, n_j + 1\}$.

Next, we construct inductively sequences (j_i), (J_i) and $(U_{n_{j_i}})$ such that for each i

(a) $j_i \leqslant J_i < j_{i+1}$,

(b) $U_{n_{j_i}} \subseteq V_{n_{j_i}}$,

(c) $\operatorname{supp}(f_{n_j}) \cap U_{n_{j_i}} = \emptyset$ for all $j > J_i$.

This would mean that if k, l are any numbers with $k > l$ then $j_k \geqslant j_{l+1} > J_l$ and so

$$\operatorname{supp}(f_{n_{j_k}}) \cap U_{n_{j_l}} = \emptyset, \tag{4.9}$$

which, combined with (4.8), would give the required result.

Base Case. Define $j_0 = 0$. If $\operatorname{supp}(f_{n_j}) \cap V_{n_{j_0}} = \emptyset$ for all $j > 0$, we set $J_0 = j_0$ and $U_{n_{j_0}} = V_{n_{j_0}}$. Otherwise we set J_0 to be any index with $J_0 > j_0$ and $\operatorname{supp}(f_{n_{J_0}}) \cap V_{n_{j_0}} \neq \emptyset$ and define $U_{n_{j_0}} = \operatorname{supp}(f_{n_{J_0}}) \cap V_{n_{j_0}}$. Note that if $j > J_0$ then

$$\operatorname{supp}(f_{n_j}) \cap U_{n_{j_0}} \subseteq \operatorname{supp}(f_{n_j}) \cap \operatorname{supp}(f_{n_{J_0}}) = \emptyset.$$

Inductive Step. Suppose that the construction has been completed up to stage i and define $j_{i+1} = J_i+1$. Similar to the above, two cases are possible. If $\operatorname{supp}(f_{n_j}) \cap V_{n_{j_{i+1}}} = \emptyset$ for all $j > j_{i+1}$, we define $J_{i+1} = j_{i+1}$ and $U_{n_{j_{i+1}}} = V_{n_{j_{i+1}}}$. Otherwise we set J_{i+1} to be any index with $J_{i+1} > j_{i+1}$ and $\operatorname{supp}(f_{n_{J_{i+1}}}) \cap V_{n_{j_{i+1}}} \neq \emptyset$ and define $U_{n_{j_{i+1}}} = \operatorname{supp}(f_{n_{J_{i+1}}}) \cap V_{n_{j_{i+1}}}$. Then the condition (c) follows from disjointness of (f_n). In either case the conditions (a)–(c) are satisfied. $\square$

We are now ready to prove the main result of the section.

Proof of Theorem 4.3.2. Suppose that an operator $T \in \mathscr{L}^{\mathscr{C}(K)}$ is noncentripetal. By Lemma 4.3.7, we can find $\varepsilon > 0$, a disjoint $(f_n) \subseteq \mathscr{C}(K)$ and a disjoint $(U_n) \subseteq \mathscr{P}(K)$ such that

$$\operatorname{range}(f_n) \subseteq [0,1], \quad \operatorname{supp}(f_m) \cap U_n = \emptyset \quad \text{and} \quad \big|[Tf_n]|_{U_n}\big| > \varepsilon \quad \forall m, n.$$

By Lemma 4.3.4, there exist $\sigma \subseteq \omega$ and $(D_n)_{n\in\sigma} \preccurlyeq (U_n)_{n\in\sigma}$ with the property that whenever $\tau \subseteq \sigma$ is such that $(f_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$ and $n \in \sigma\backslash\tau$, we have

$$\big|[T(\vee_{n\in\tau} f_n)]|_{D_n}\big| \leqslant \frac{\varepsilon}{3}. \tag{4.10}$$

Applying property (K) to $(f_n)_{n\in\sigma}$ and $(D_n)_{n\in\sigma}$, we can find $\tau \subseteq \sigma$ such that

(i) $(f_n)_{n\in\tau}$ has a supremum in $\mathscr{C}(K)$,

(ii) $\overline{\bigcup_{n\in\tau} D_n} \cap \overline{\bigcup_{n\in\sigma\setminus\tau} D_n} \neq \emptyset$.

Note that (4.10) implies that

$$\overline{\bigcup_{n\in\sigma\setminus\tau} D_n} \subseteq (T(\vee_{n\in\tau} f_n))^{-1}([-\varepsilon/3, \varepsilon/3]). \tag{4.11}$$

On the other hand, let $n \in \tau$. If we define $\tau' = \tau\setminus\{n\}$, then $(f_n)_{n\in\tau'}$ has a supremum in $\mathscr{C}(K)$ and so, by (4.10),

$$\left|[T(\vee_{n\in\tau'} f_n)]|_{D_n}\right| \leqslant \frac{\varepsilon}{3}.$$

However, we have

$$\left|[Tf_n]|_{D_n}\right| > \varepsilon$$

and so

$$\left|[T(\vee_{n\in\tau} f_n)]|_{D_n}\right| = \left|[T(\vee_{n\in\tau'} f_n)]|_{D_n} + [Tf_n]|_{D_n}\right| > \varepsilon - \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

implying that

$$\overline{\bigcup_{n\in\tau} D_n} \subseteq (T(\vee_{n\in\tau} f_n))^{-1}([-\infty, -2\varepsilon/3] \cup [2\varepsilon/3, \infty]). \tag{4.12}$$

Combination of (4.11) and (4.12) gives that

$$\overline{\bigcup_{n\in\tau} D_n} \cap \overline{\bigcup_{n\in\sigma\setminus\tau} D_n} = \emptyset$$

leading to a contradiction to the condition (ii). Hence our original assumption was wrong and every operator on $\mathscr{C}(K)$ is indeed centripetal. □

4.4 Preservation of suprema and forbidden splittings

Definition 4.4.1. Suppose that K and L are topological spaces such that there exists a continuous surjection $\rho_K^L \colon L \twoheadrightarrow K$.

- Let $(f_n) \subseteq \mathscr{C}(K)$ be a disjoint sequence with $\mathrm{range}(f_n) \subseteq [0,1]$ for each n and suppose also that (f_n) has a supremum in $\mathscr{C}(K)$. We say that *the supremum of* (f_n) *is preserved in* $\mathscr{C}(L)$ if
  - $(f_n^{[L]})$ has a supremum in $\mathscr{C}(L)$, and

– $\bigvee_{n\in\omega}(f_n^{[L]}) = (\bigvee_{n\in\omega} f_n)^{[L]}$.

- Suppose that $(\mathcal{P}, \mathcal{Q}) \subseteq \mathscr{P}(K) \times \mathscr{P}(K)$ is a forbidden splitting in K. We say that $(\mathcal{P}, \mathcal{Q})$ *is preserved in* L if $\big(\mathcal{P}^{[L]}, \mathcal{Q}^{[L]}\big)$ is a forbidden splitting in L.

Notation 4.4.2. To avoid too many brackets, if the supremum of $(f_n) \subseteq K$ is preserved in L, we will write $\bigvee_{n\in\omega} f_n^{[L]}$ to denote both $\bigvee_{n\in\omega}(f_n^{[L]})$ and $(\bigvee_{n\in\omega} f_n)^{[L]}$.

As in the zero-dimensional case, we now proceed to establish a framework for the transfinite induction argument in section 4.5.

Proposition 4.4.3. *Let α be an ordinal with $1 \leqslant \alpha < 2^\omega$ and consider a topological space K such that*

(a) K is (compact, Hausdorff and) connected,

(b) $\text{weight}(K) = \kappa < 2^\omega$ so that we can fix a basis $\mathscr{B}$ for K with $|\mathscr{B}| = \kappa$,

(c) $\text{density}(\mathscr{C}(K)) = d < 2^\omega$,

(d) there exists a continuous irreducible surjection $\phi\colon K \twoheadrightarrow [0,1]^\alpha$.

Suppose also that we are given

(e) a disjoint sequence $(f_n) \subseteq \mathscr{C}(K)$ with $\text{range}(f_n) \subseteq [0,1]$ for each n,

(f) a disjoint sequence $(g_n) \subseteq \mathscr{C}(K)$ with $\text{range}(g_n) \subseteq [0,1]$ and such that (g_n) has a supremum in $\mathscr{C}(K)$,

(g) a disjoint sequence (U_n) of nonempty open subsets of K with $\text{supp}(f_m) \cap U_n = \emptyset$ for all m, n,

(h) an ordinal $\nu < 2^\omega$ and a transfinite sequence $((\mathcal{P}_\beta, \mathcal{Q}_\beta))_{\beta<\nu}$ of forbidden splittings in K.

As before, for each $\tau \subseteq \omega$ we define

$$K(\tau) = \textit{the extension of } K \textit{ by } (f_n)_{n\in\tau}.$$

Then

(i) $K(\tau)$ *is compact and Hausdorff,*

(ii) $\text{weight}(K(\tau)) < 2^\omega$ *and, in fact, there exists a basis* $\mathscr{B}(\tau)$ *of* $K(\tau)$ *with* $|\mathscr{B}(\tau)| < 2^\omega$ *and* $\mathscr{B}^{[K(\tau)]} \subseteq \mathscr{B}(\tau)$*,*

(iii) $\text{density}(\mathscr{C}(K(\tau))) < 2^\omega$*,*

(iv) the supremum of $(g_n)_{n\in\omega}$ *is preserved in* $K(\tau)$*,*

(v) the function $f_\tau^{[K(\tau)]}$ *defined in* (4.1) *is a supremum of* $(f_n^{[K(\tau)]})_{n\in\tau}$ *in* $\mathscr{C}(K(\tau))$*.*

In addition to the above, there exists an infinite $\sigma \subseteq \omega$ *such that for any* $\tau \subseteq \sigma$

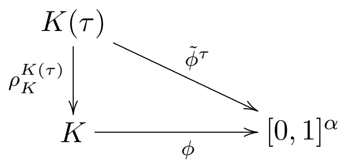
(vi) $K(\tau)$ *is a strong extension of* K*,*

and consequently

(vii) $K(\tau)$ *is connected,*

(viii) $\rho_K^{K(\tau)}$ *is a continuous irreducible surjection,*

(ix) there exists a continuous irreducible surjection $\tilde{\phi}^\tau \colon K(\tau) \twoheadrightarrow [0,1]^\alpha$ *such that*



is a commutative diagram,

Furthermore, with σ *as above we can find* $\theta \subseteq \sigma$ *such that whenever* $\tau \subseteq \theta$*,*

(x) all splittings $(\mathcal{P}_\beta, \mathcal{Q}_\beta)$ *are preserved in* $K(\tau)$*.*

Finally, with θ *as above, there exists* $\tau \subseteq \theta$ *such that*

(xi) the pair $\left((U_n^{[K(\tau)]})_{n\in\tau}, (U_n^{[K(\tau)]})_{n\notin\tau}\right)$ *cannot be split in* $K(\tau)$*.*

Proof. (i) $K(\tau)$ is clearly Hausdorff and, being a closed subset of $K \times [0,1]$, compact.

(ii) Since $K(\tau) \subseteq K \times [0,1]$, we have

$$\text{weight}(K(\tau)) = \max\{\text{weight}(K), \text{weight}([0,1])\} < 2^\omega,$$

and so there exists a basis $\underline{\mathscr{B}}(\tau)$ for $K(\tau)$ with $|\underline{\mathscr{B}}(\tau)| < 2^\omega$. Note that $U^{[K(\tau)]}$ is open in $K(\tau)$ whenever $U \in \mathscr{B}$, thus we can take $\mathscr{B}(\tau)$ to be the basis generated by $\mathscr{B}^{[K(\tau)]}$ and $\underline{\mathscr{B}}(\tau)$. Then

$$|\mathscr{B}(\tau)| = \max\{|\mathscr{B}^{[K(\tau)]}|, |\underline{\mathscr{B}}(\tau)|\} < 2^\omega$$

as required.

(iii) By Corollary 4.2.18,

$$\text{density}(\mathscr{C}(K(\tau))) = \text{density}(\mathscr{C}(K)) < 2^\omega.$$

(iv) Note that whenever $g \in \mathscr{C}(K)$, we have

$$\begin{aligned}
\Delta\left(g^{[K(\tau)]}, (g_n^{[K(\tau)]})_{n\in\omega}\right) &= \left\{(x,t) \in K(\tau) : \sum_{n\in\omega} g_n\left(\rho_K^{K(\tau)}(x,t)\right) \neq g\left(\rho_K^{K(\tau)}(x,t)\right)\right\} \\
&= \left\{(x,t) \in K(\tau) : \rho_K^{K(\tau)}(x,t) \in \Delta(g, (g_n)_{n\in\omega})\right\} \\
&= \left(\rho_K^{K(\tau)}\right)^{-1}\left(\Delta(g, (g_n)_{n\in\omega})\right).
\end{aligned}$$

By Lemma 4.2.4, $\Delta(\vee_{n\in\omega} g_n, (g_n)_{n\in\omega})$ is nowhere dense, hence, by Lemma 4.2.22, so is $(\rho_K^{K(\tau)})^{-1}(\Delta(\vee_{n\in\omega} g_n, (g_n)_{n\in\omega}))$. Applying Lemma 4.2.4 again, this means that $g^{[K(\tau)]}$ is a supremum of $(g_n^{[K(\tau)]})$ in $\mathscr{C}(K(\tau))$ as required.

(v) This is Theorem 4.2.15.

(vi) This is Theorem 4.2.23.

(vii) This is Theorem 4.2.19.

(viii) This is Corollary 4.2.21.

(ix) We define

$$\tilde{\phi}^\tau = \phi\rho_K^{K(\tau)}.$$

Since both $\rho_K^{K(\tau)}$ and ϕ are continuous irreducible surjections, so is $\tilde{\phi}^\tau$.

(x) Let $\{\theta_\xi\}_{\xi<2^\omega}$ be an almost disjoint family of subsets of σ (for existence of such a family see [Kun80, p. 48, Theorem 1.3]) and suppose, for a contradiction, that the statement is false. Then for every $\xi < 2^\omega$ there exist $\tau_\xi \subseteq \theta_\xi$ and $\beta < \nu$ such that $\left(\mathcal{P}_\beta^{[K(\tau_\xi)]}, \mathcal{Q}_\beta^{[K(\tau_\xi)]}\right)$

is split in $K(\tau_\xi)$. In particular, there exists a continuous function $F_\xi\colon K(\tau_\xi) \to [0,1]$ such that

$$\bigcup \mathcal{P}_\beta^{[K(\tau_\xi)]} \subseteq (F_\xi)^{-1}(\{0\}), \quad \bigcup \mathcal{Q}_\beta^{[K(\tau_\xi)]} \subseteq (F_\xi)^{-1}(\{1\}).$$

Let D be a dense subset of $\mathscr{C}(K)$ of cardinality d. Corollary 4.2.18 implies that there exist $k \in \omega$ and $g_0, \dots, g_k \in D$ such that if we define the function G_ξ by setting

$$G_\xi = g_0^{[K(\tau_\xi)]} + g_1^{[K(\tau_\xi)]} \cdot f_{\tau_\xi}^{[K(\tau_\xi)]} + \dots + g_k^{[K(\tau_\xi)]} \cdot \left(f_{\tau_\xi}^{[K(\tau_\xi)]}\right)^k,$$

then

$$\|F_\xi - G_\xi\|_\infty < \frac{1}{4},$$

and so

$$\bigcup \mathcal{P}_\beta^{[K(\tau_\xi)]} \subseteq G_\xi^{-1}([-1/4, 1/4]), \quad \bigcup \mathcal{Q}_\beta^{[K(\tau_\xi)]} \subseteq G_\xi^{-1}([3/4, 5/4]). \tag{4.13}$$

However, there are at most $\max\{\omega, d, |\nu|\} < 2^\omega$ possibilities for picking a tuple $(k, g_0, \dots, g_k, \beta)$ while there are 2^ω choices for picking ξ. As a result, there must exist distinct $\xi, \eta < 2^\omega$ for which the same choice of $(k, g_0, \dots, g_k, \beta)$ are made. That is, in addition to (4.13) we have

$$\bigcup \mathcal{P}_\beta^{[K(\tau_\eta)]} \subseteq G_\eta^{-1}([-1/4, 1/4]), \quad \bigcup \mathcal{Q}_\beta^{[K(\tau_\eta)]} \subseteq G_\eta^{-1}([3/4, 5/4]), \tag{4.14}$$

where

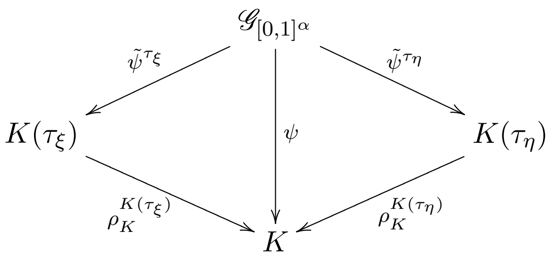
$$G_\eta = g_0^{[K(\tau_\eta)]} + g_1^{[K(\tau_\eta)]} \cdot f_{\tau_\eta}^{[K(\tau_\eta)]} + \dots + g_k^{[K(\tau_\eta)]} \cdot \left(f_{\tau_\eta}^{[K(\tau_\eta)]}\right)^k.$$

Following the algorithm from the previous chapter, [Hay81] or [Kos04], it would now be logical to consider the product of G_ξ and G_η (since τ_ξ and τ_η are almost disjoint). However, in order for this to be well-defined, we need to move to a bigger space which "contains" both $K(\tau_\xi)$ and $K(\tau_\eta)$. The Gleason space $\mathscr{G}_{[0,1]^\alpha}$ turns out to be a suitable choice.

Note that, by projectivity of $\mathscr{G}_{[0,1]^\alpha}$, there exists a continuous $\psi\colon \mathscr{G}_{[0,1]^\alpha} \to K$ with

$$\gamma_{[0,1]^\alpha} = \phi\psi.$$

The same argument applied to the pairs $(\psi, \rho_K^{K(\tau_\xi)})$ and $(\psi, \rho_K^{K(\tau_\eta)})$ gives rise to continuous $\tilde{\psi}^{\tau_\xi}\colon \mathscr{G}_{[0,1]^\alpha} \to K(\tau_\xi)$ and $\tilde{\psi}^{\tau_\eta}\colon \mathscr{G}_{[0,1]^\alpha} \to K(\tau_\eta)$ such that all parts of the diagram



commute. Finally, since $\gamma_{[0,1]^\alpha}$, $\rho_K^{K(\tau_\xi)}$ and $\rho_K^{K(\tau_\eta)}$ are continuous irreducible surjections, Corollary 4.2.10 implies that so are all the functions involved in the above diagram.

Thus, as before, we can define liftings of functions from $\mathscr{C}(K)$, $\mathscr{C}(K(\tau_\xi))$ and $\mathscr{C}(K(\tau_\xi))$ to $\mathscr{C}(\mathscr{G}_{[0,1]^\alpha})$. Specifically, if $f \in \mathscr{C}(K)$, $g \in \mathscr{C}(K(\tau_\xi))$ and $h \in \mathscr{C}(K(\tau_\eta))$, we define

$$f^{[\mathscr{G}_{[0,1]^\alpha}]} = f \circ \psi, \quad g^{[\mathscr{G}_{[0,1]^\alpha}]} = g \circ \tilde{\psi}^{\tau_\xi}, \quad h^{[\mathscr{G}_{[0,1]^\alpha}]} = h \circ \tilde{\psi}^{\tau_\eta}.$$

In the same fashion, we define liftings of subsets of K, $K(\tau_\xi)$ and $K(\tau_\eta)$ to $\mathscr{G}_{[0,1]^\alpha}$.

In particular, using (4.13), we see that

$$\begin{aligned} \bigcup \mathcal{P}_\beta^{[\mathscr{G}_{[0,1]^\alpha}]} &= \left(\tilde{\psi}^{\tau_\xi}\right)^{-1} \left(\bigcup \mathcal{P}_\beta^{[K(\tau_\xi)]}\right) \\ &\subseteq \left(\tilde{\psi}^{\tau_\xi}\right)^{-1} \left(G_\xi^{-1}([-1/4, 1/4])\right) \\ &= \left(G_\xi \circ \tilde{\psi}^{\tau_\xi}\right)^{-1} ([-1/4, 1/4]) \\ &= \left(G_\xi^{[\mathscr{G}_{[0,1]^\alpha}]}\right)^{-1} ([-1/4, 1/4]). \end{aligned}$$

Applying the same argument to (4.14), we get

$$\bigcup \mathcal{P}_\beta^{[\mathscr{G}_{[0,1]^\alpha}]} \subseteq \left(G_\eta^{[\mathscr{G}_{[0,1]^\alpha}]}\right)^{-1} ([-1/4, 1/4]),$$

and thus

$$\bigcup \mathcal{P}_\beta^{[\mathscr{G}_{[0,1]^\alpha}]} \subseteq \left(G_\xi^{[\mathscr{G}_{[0,1]^\alpha}]} \cdot G_\eta^{[\mathscr{G}_{[0,1]^\alpha}]}\right)^{-1} ([-1/16, 1/16]). \tag{4.15}$$

Similarly,

$$\bigcup \mathcal{Q}_\beta^{[\mathscr{G}_{[0,1]^\alpha}]} \subseteq \left(G_\xi^{[\mathscr{G}_{[0,1]^\alpha}]} \cdot G_\eta^{[\mathscr{G}_{[0,1]^\alpha}]}\right)^{-1} ([9/16, 25/16]). \tag{4.16}$$

Note that $G_\xi^{[\mathscr{G}_{[0,1]^\alpha}]} \cdot G_\eta^{[\mathscr{G}_{[0,1]^\alpha}]} \in \mathscr{C}(K)^{[\mathscr{G}_{[0,1]^\alpha}]}$. Indeed, for each i, j in $\{0, \ldots, k\}$ let

$$F_{ij} = \left(\left(f_{\tau_\xi}^{[K(\tau_\xi)]}\right)^{[\mathscr{G}_{[0,1]^\alpha}]}\right)^i \cdot \left(\left(f_{\tau_\eta}^{[K(\tau_\eta)]}\right)^{[\mathscr{G}_{[0,1]^\alpha}]}\right)^j \in \mathscr{C}(\mathscr{G}_{[0,1]^\alpha}).$$

Take any $x \in \psi^{-1}(\mathcal{D}(\tau_\xi) \cap \mathcal{D}(\tau_\eta))$. Lemma 4.2.20 implies that

$$\left(\rho_K^{K(\tau_\xi)}\right)^{-1} (\{\psi(x)\}) = \left\{ \left(\psi(x), \sum_{n \in \tau_\xi} f_n(\psi(x)) \right) \right\}.$$

But $\psi = \rho_K^{K(\tau_\xi)} \circ \tilde{\psi}^{\tau_\xi}$ which implies that

$$\tilde{\psi}^{\tau_\xi}(x) = \left(\psi(x), \sum_{n \in \tau_\xi} (f_n \circ \psi)(x) \right)$$

and so

$$\left(f_{\tau_\xi}^{[K(\tau_\xi)]}\right)^{[\mathscr{G}_{[0,1]^\alpha}]}(x) = f_{\tau_\xi}^{[K(\tau_\xi)]}\left(\psi(x), \sum_{n\in\tau_\xi}(f_n\circ\psi)(x)\right) = \sum_{n\in\tau_\xi}(f_n\circ\psi)(x).$$

Similarly,

$$\left(f_{\tau_\eta}^{[K(\tau_\eta)]}\right)^{[\mathscr{G}_{[0,1]^\alpha}]}(x) = \sum_{n\in\tau_\eta}(f_n\circ\psi)(x).$$

Putting the last two expressions together and using disjointness of (f_n), we conclude that

$$F_{ij} = \left(\sum_{n\in\tau_\xi} f_n^i\circ\psi\right)\cdot\left(\sum_{n\in\tau_\eta} f_n^j\circ\psi\right) = \left(\sum_{n\in\tau_\xi\cap\tau_\eta} f_n^{i+j}\right)\circ\psi \quad \text{on } \psi^{-1}\left(\mathcal{D}(\tau_\xi)\cap\mathcal{D}(\tau_\eta)\right).$$

However, Lemma 4.2.6 tells us that each of $\mathcal{D}(\tau_\xi)$ and $\mathcal{D}(\tau_\eta)$ is an open dense subset of K, which means that so is their intersection. Irreducibility of ψ and Lemma 4.2.12 imply that $\psi^{-1}\left(\mathcal{D}(\tau_\xi)\cap\mathcal{D}(\tau_\eta)\right)$ is an open dense subset of $\mathscr{G}_{[0,1]^\alpha}$. Hence, by continuity of all functions involved in the above formula, we can deduce that

$$F_{ij} = \left(\sum_{n\in\tau_\xi\cap\tau_\eta} f_n^{i+j}\right)\circ\psi = \left(\sum_{n\in\tau_\xi\cap\tau_\eta} f_n^{i+j}\right)^{[\mathscr{G}_{[0,1]^\alpha}]}.$$

In particular, since $\tau_\xi\cap\tau_\eta$ is finite, $F_{ij}\in\mathscr{C}(K)^{[\mathscr{G}_{[0,1]^\alpha}]}$.

Going back to the definitions of G_ξ and G_η, note that

$$G_\xi^{[\mathscr{G}_{[0,1]^\alpha}]}\cdot G_\eta^{[\mathscr{G}_{[0,1]^\alpha}]} = \sum_{l=0}^{2k}\sum_{\substack{0\leqslant i,j\leqslant k\\ i+j=l}} g_i^{[\mathscr{G}_{[0,1]^\alpha}]} g_j^{[\mathscr{G}_{[0,1]^\alpha}]} F_{ij}.$$

Thus

$$G_\xi^{[\mathscr{G}_{[0,1]^\alpha}]}\cdot G_\eta^{[\mathscr{G}_{[0,1]^\alpha}]} = G^{[\mathscr{G}_{[0,1]^\alpha}]}$$

for some $G\in\mathscr{C}(K)$. Consequently, (4.15) may be rewritten as

$$\bigcup\mathcal{P}_\beta^{[\mathscr{G}_{[0,1]^\alpha}]}\subseteq\left(G^{[\mathscr{G}_{[0,1]^\alpha}]}\right)^{-1}([-1/16,1/16]).$$

Equivalently,

$$\psi^{-1}(\bigcup\mathcal{P}_\beta)\subseteq(G\circ\psi)^{-1}([-1/16,1/16]) = \psi^{-1}\left(G^{-1}([-1/16,1/16])\right).$$

However, ψ is a surjection, hence $\psi\psi^{-1}=\mathrm{I}_K$. Thus the last equation implies that

$$\bigcup\mathcal{P}_\beta\subseteq G^{-1}([-1/16,1/16]).$$

Applying the same argument to (4.16), we also get

$$\bigcup \mathcal{Q}_\beta \subseteq G^{-1}([9/16, 25/16])$$

which, contrary to our initial assumption, means that $(\mathcal{P}_\beta, \mathcal{Q}_\beta)$ is split in K.

(xi) First of all, we show that whenever the pair $((U_n)_{n\in\tau}, (U_n)_{n\notin\tau})$ is forbidden in K for some $\tau \subseteq \sigma$, it remains forbidden in $K(\tau)$.

Indeed, suppose that $\left((U_n^{[K(\tau)]})_{n\in\tau}, (U_n^{[K(\tau)]})_{n\notin\tau}\right)$ is split in $K(\tau)$. Using the same argument as in (4.13) and (4.14), we can find $k \in \omega$ and $h_0, \ldots, h_k \in \mathscr{C}(K)$ such that

$$\bigcup_{n\in\tau} U_n^{[K(\tau)]} \subseteq H^{-1}([-1/4, 1/4]), \quad \bigcup_{n\notin\tau} U_n^{[K(\tau)]} \subseteq H^{-1}([3/4, 5/4]),$$

where H is defined by setting

$$H = h_0^{[K(\tau)]} + h_1^{[K(\tau)]} \cdot f_\tau^{[K(\tau)]} + \cdots + h_k^{[K(\tau)]} \cdot \left(f_\tau^{[K(\tau)]}\right)^k.$$

Note that for each n we have

$$f_\tau^{[K(\tau)]}\left(U_n^{[K(\tau)]}\right) = 0. \tag{4.17}$$

This is because if $(x, t) \in U_n^{[K(\tau)]} \cap \operatorname{supp}(f_\tau^{[K(\tau)]})$, Lemma 4.2.2 implies that there exist $(x', t') \in U_n^{[K(\tau)]}$ and $m \in \tau$ with

$$0 \neq f_m^{[K(\tau)]}((x', t')) = f_m\left(\rho_K^{K(\tau)}((x', t'))\right) = f_m(x').$$

This, in turn, means that $x' \in \operatorname{supp}(f_m) \cap U_n$ which contradicts our initial assumption.

Equation (4.17) now implies that

$$\begin{aligned}
[-1/4, 1/4] \supseteq H\left(\bigcup_{n\in\tau} U_n^{[K(\tau)]}\right) &= h_0^{[K(\tau)]}\left(\bigcup_{n\in\tau} U_n^{[K(\tau)]}\right) \\
&= \left(h_0 \circ \rho_K^{K(\tau)}\right)\left(\left(\rho_K^{K(\tau)}\right)^{-1}\left(\bigcup_{n\in\tau} U_n\right)\right) \\
&= h_0\left(\bigcup_{n\in\tau} U_n\right),
\end{aligned}$$

where the last equality follows from surjectivity of $\rho_K^{K(\tau)}$. Thus

$$\bigcup_{n\in\tau} U_n \subseteq h_0^{-1}([-1/4, 1/4]).$$

Similarly,

$$\bigcup_{n\notin\tau} U_n \subseteq h_0^{-1}([3/4,5/4]),$$

and so $((U_n)_{n\in\tau},(U_n)_{n\notin\tau})$ is split in K as required.

To complete the proof, pick μ with $|\mu|=\kappa$ and consider a basis $\mathscr{B}=\{B_j : j<\mu\}$ of K. Note that if $((U_n)_{n\in\tau},(U_n)_{n\notin\tau})$ is split in K then, since K is compact and Hausdorff, there exists a finite $\Lambda(\tau)\subseteq\mu$ with

$$\overline{\bigcup_{n\in\tau} U_n} \subseteq \bigcup_{n\in\Lambda(\tau)} B_n, \quad \overline{\bigcup_{n\notin\tau} U_n} \cap \bigcup_{n\in\Lambda(\tau)} B_n = \emptyset.$$

Let now τ and τ' be distinct subsets of θ such that both $((U_n)_{n\in\tau},(U_n)_{n\notin\tau})$ and $\big((U_n)_{n\in\tau'},(U_n)_{n\notin\tau'}\big)$ are split in K. Without loss of generality there exists $m\in\tau\backslash\tau'$. Then

$$\overline{U_m} \subseteq \overline{\bigcup_{n\in\tau} U_n} \subseteq \bigcup_{n\in\Lambda(\tau)} B_n,$$

while

$$\overline{U_m} \cap \bigcup_{n\in\Lambda(\tau')} B_n \subseteq \overline{\bigcup_{n\notin\tau'} U_n} \cap \bigcup_{n\in\Lambda(\tau')} B_n = \emptyset.$$

Since $U_m\neq\emptyset$, this means that $\Lambda(\tau)\neq\Lambda(\tau')$.

However, there are only $\kappa<2^\omega$ choices for picking a finite subset $\Lambda(\tau)\subseteq\mu$ versus 2^ω choices for picking a subset τ of θ. This means that there must exist $\tau\subseteq\theta$ for which $\Lambda(\tau)$ cannot be found or, equivalently, the splitting $((U_n)_{n\in\tau},(U_n)_{n\notin\tau})$ is forbidden in K. As shown earlier, it remains forbidden in $K(\tau)$. □

Proposition 4.4.4. *Let α be an ordinal with $1\leqslant\alpha<2^\omega$ and consider a topological space K such that*

(a) K is (compact, Hausdorff and) connected,

(b) $\mathrm{weight}(K)=\kappa<2^\omega$ *so that we can fix a basis $\mathscr{B}$ for K with $|\mathscr{B}|<\kappa$,*

(c) $\mathrm{density}(\mathscr{C}(K))<2^\omega$*,*

(d) there exists a continuous irreducible surjection $\phi\colon K\twoheadrightarrow[0,1]^\alpha$.

Suppose also that we are given

(e) *a disjoint sequence* $(g_n) \subseteq \mathscr{C}(K)$ *such that* $\text{range}(g_n) \subseteq [0,1]$ *for each* n *and* (g_n) *has a supremum in* $\mathscr{C}(K)$*,*

(f) *a forbidden splitting* $(\mathcal{P}, \mathcal{Q})$ *in* K*.*

Define

$$L = K \times [0,1].$$

Then

(i) L *is compact Hausdorff and connected,*

(ii) $\text{weight}(L) < 2^\omega$ *and moreover there exists a basis* $\mathscr{E}$ *for* L *with* $|\mathscr{E}| < 2^\omega$ *and* $\mathscr{B}^{[L]} \subseteq \mathscr{E}$*,*

(iii) $\text{density}(\mathscr{C}(L)) < 2^\omega$*,*

(iv) *there exists a continuous surjection* $\rho^L_K \colon L \twoheadrightarrow K$*,*

(v) *the supremum of* (g_n) *is preserved in* $\mathscr{C}(L)$*,*

(vi) *the splitting* $(\mathcal{P}, \mathcal{Q})$ *is preserved in* L*,*

(vii) *each of* $(\mathcal{P} \times [0,1/3], \mathcal{Q} \times [0,1/3])$*,* $(\mathcal{P} \times [2/3,1], \mathcal{Q} \times [2/3,1])$ *forms a forbidden splitting in* L*,*

(viii) *there exists an irreducible surjection* $\phi' \colon L \twoheadrightarrow [0,1]^{\alpha+1}$ *such that the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\phi'} & [0,1]^{\alpha+1} \\ {\scriptstyle \rho^L_K}\downarrow & & \downarrow{\scriptstyle \pi^{\alpha+1}_\alpha} \\ K & \xrightarrow[\phi]{} & [0,1]^\alpha \end{array}$$

is commutative.

Proof. (i) This is immediate as both K and $[0,1]$ are compact Hausdorff and connected.

(ii) Let $\mathscr{B}'$ be a countable basis for $[0,1]$. Define $\mathscr{E}$ to be the basis generated by $\{B \times [0,1] : B \in \mathscr{B}\}$ and $\{K \times B' : B' \in \mathscr{B}'\}$. Then

$$|\mathscr{E}| = \max\{|\mathscr{B}|, |\mathscr{B}'|\} < 2^\omega$$

and clearly $\mathscr{B}^{[L]} = \mathscr{B} \times [0,1] \subseteq \mathscr{E}$.

(iii) By the Stone–Weierstrass Theorem [DS58, IV.6.15, Theorem 16],

$$\mathscr{C}(L) = \overline{\langle \mathcal{F}^{[L]}, \mathcal{G}^{[L]} \rangle},$$

whenever $\mathcal{F}$ is a dense subset in $\mathscr{C}(K)$ and $\mathcal{G}$ is a dense subset of $\mathscr{C}([0,1])$. Thus

$$\text{density}(\mathscr{C}(L)) = \max\{\text{density}(\mathscr{C}(K)), \text{density}(\mathscr{C}([0,1]))\} < 2^{\omega}.$$

(iv) For each $(x,t) \in L$ we define

$$\rho_K^L((x,t)) = x.$$

(v) By Lemma 4.2.4, the set $\Delta(\vee_{n\in\omega} g_n, (g_n)_{n\in\omega})$ is nowhere dense. But then so is

$$\Delta((\vee_{n\in\omega} g_n)^{[L]}, (g_n^{[L]})_{n\in\omega}) = \Delta(\vee_{n\in\omega} g_n, (g_n)_{n\in\omega}) \times [0,1].$$

Lemma 4.2.4 now yields that $(\vee_{n\in\omega} g_n)^{[L]}$ is a supremum of $(g_n^{[L]})_{n\in\omega}$ in $\mathscr{C}(L)$.

(vi) Note that if $(\mathcal{P}, \mathcal{Q})$ is a forbidden splitting, then

$$\overline{\bigcup \mathcal{P}^{[L]}} \cap \overline{\bigcup \mathcal{Q}^{[L]}} = \overline{\bigcup \mathcal{P} \times [0,1]} \cap \overline{\bigcup \mathcal{Q} \times [0,1]} = \left(\overline{\bigcup \mathcal{P}} \cap \overline{\bigcup \mathcal{Q}} \right) \times [0,1] \neq \emptyset,$$

and so $\left(\mathcal{P}^{[L]}, \mathcal{Q}^{[L]}\right)$ cannot be split in L.

(vii) The proof is precisely the same as in (vi): since $\overline{\bigcup \mathcal{P}} \cap \overline{\bigcup \mathcal{Q}} \neq \emptyset$,

$$\overline{\bigcup \mathcal{P} \times [0,1/3]} \cap \overline{\bigcup \mathcal{Q} \times [0,1/3]} = \left(\overline{\bigcup \mathcal{P}} \cap \overline{\bigcup \mathcal{Q}} \right) \times [0,1/3] \neq \emptyset,$$

and similar for $(\mathcal{P} \times [2/3,1], \mathcal{Q} \times [2/3,1])$.

(viii) We define $\phi' \colon L \to [0,1]^{\alpha+1}$ by setting

$$\phi'(x,t) = (\phi(x), t) \quad \forall (x,t) \in L.$$

Since ϕ and $\mathrm{I}_{[0,1]}$ are continuous irreducible surjections, Lemma 4.2.11 implies that so is ϕ'. Finally, for any $(x,t) \in L$ we have

$$\begin{aligned} \pi_{\alpha}^{\alpha+1}(\phi'((x,t))) &= \pi_{\alpha}^{\alpha+1}((\phi(x), t)) \\ &= \phi(x) \\ &= \phi\left(\rho_K^L((x,t))\right), \end{aligned}$$

and so

$$\pi_\alpha^{\alpha+1} \circ \phi' = \phi \circ \rho_K^L.$$

□

As mentioned in the introduction, the topological analogue of the union of Boolean algebras is the inverse limit of an inverse system of topological spaces. For more information on inverse limits and related topics we refer the reader to [Eng89, Section 2.5].

Proposition 4.4.5. *Let $\alpha \leqslant 2^\omega$ be a limit ordinal and suppose that we have a transfinite sequence $(K_\beta)_{\beta<\alpha}$ of topological spaces such that*

(a) each K_β is (compact, Hausdorff and) connected,

(b) $\mathrm{weight}(K_\beta) < 2^\omega$ so that we can fix a basis $\mathscr{B}_\beta$ of K_β with $|\mathscr{B}_\beta| < 2^\omega$,

(c) $\mathrm{density}(\mathscr{C}(K_\beta)) < 2^\omega$ so that we can fix a dense subset D_β of $\mathscr{C}(K_\beta)$ with $|D_\beta| < 2^\omega$; without loss of generality we may also assume that $\chi_{K_\beta} \in D_\beta$,

(d) for any $\gamma \leqslant \beta < \alpha$ there exists a continuous surjection $\rho_{K_\gamma}^{K_\beta} \colon K_\beta \twoheadrightarrow K_\gamma$ such that the family

$$S = \left\{K_\beta, \rho_{K_\gamma}^{K_\beta}, \alpha\right\}$$

forms an inverse system, that is, for every $\delta \leqslant \gamma \leqslant \beta < \alpha$ we have

$$\rho_{K_\delta}^{K_\gamma} \rho_{K_\gamma}^{K_\beta} = \rho_{K_\delta}^{K_\beta} \quad \text{and} \quad \rho_{K_\beta}^{K_\beta} = \mathrm{I}_{K_\beta},$$

(e) for each $\beta < \alpha$ there exists a continuous irreducible surjection $\phi_\beta \colon K_\beta \twoheadrightarrow [0,1]^\beta$ such that for any $\gamma \leqslant \beta < \alpha$ the diagram

$$\begin{array}{ccc} K_\beta & \xrightarrow{\ \phi_\beta\ } & [0,1]^\beta \\ {\scriptstyle \rho_{K_\gamma}^{K_\beta}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\gamma^\beta} \\ K_\gamma & \xrightarrow[\ \phi_\gamma\ ]{} & [0,1]^\gamma \end{array}$$

is commutative.

Fix now some $\beta_0 < \alpha$ and suppose that we are also given

(f) a disjoint sequence $(g_n) \subseteq \mathscr{C}(K_{\beta_0})$ with $\mathrm{range}(g_n) \subseteq [0,1]$ for each n and such that a supremum of (g_n) exists in $\mathscr{C}(K_{\beta_0})$ and is preserved in $\mathscr{C}(K_\beta)$ whenever $\beta_0 \leqslant \beta < \alpha$,

(g) *a splitting* $(\mathcal{P},\mathcal{Q})$ *which is forbidden in* K_{β_0} *and is preserved in* K_β *whenever* $\beta_0 \leqslant \beta < \alpha$.

Define

$$K_\alpha = \varprojlim_{\beta<\alpha} K_\beta$$

and for each $\beta < \alpha$ *set*

$$\rho^{K_\alpha}_{K_\beta} \colon K_\alpha \to K_\beta, \quad \rho^{K_\alpha}_{K_\beta}((x_\beta)_{\beta<\alpha}) = x_\beta \quad \forall (x_\beta)_{\beta<\alpha} \in K_\alpha.$$

Then

(i) K_α *is compact Hausdorff and connected,*

(ii) *for any* $\beta \leqslant \alpha$ *the map* $\rho^{K_\alpha}_{K_\beta}$ *is continuous and onto and, in addition,*

$$\rho^{K_\beta}_{K_\gamma}\rho^{K_\alpha}_{K_\beta} = \rho^{K_\alpha}_{K_\gamma}$$

whenever $\gamma \leqslant \beta \leqslant \alpha$,

(iii) *the set*

$$\mathscr{B} = \bigcup_{\beta<\alpha} \left\{ B^{[K_\alpha]} : B \in \mathscr{B}_\beta \right\} \tag{4.18}$$

is a basis of K_α*; in particular, if* $\alpha < 2^\omega$*, then* $\mathrm{weight}(K_\alpha) < 2^\omega$,

(iv) *there exists an irreducible surjection* $\phi_\alpha \colon K_\alpha \twoheadrightarrow [0,1]^\alpha$ *such that the diagram*

$$\begin{array}{ccc} K_\alpha & \xrightarrow{\ \phi_\alpha\ } & [0,1]^\alpha \\ {\scriptstyle \rho^{K_\alpha}_{K_\beta}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^\alpha_\beta} \\ K_\beta & \xrightarrow[\ \phi_\beta\ ]{} & [0,1]^\beta \end{array}$$

is commutative for every $\beta < \alpha$,

(v) *the algebra*

$$D = \left\langle \{ D^{[K_\alpha]}_\beta : \beta < \alpha \} \right\rangle \tag{4.19}$$

is dense in $\mathscr{C}(K_\alpha)$*; in particular, if* $\alpha < 2^\omega$*, then* $\mathrm{density}(\mathscr{C}(K_\alpha)) < 2^\omega$,

(vi) *the supremum of* (g_n) *is preserved in* $\mathscr{C}(K_\alpha)$,

(vii) *the splitting* $(\mathcal{P},\mathcal{Q})$ *is preserved in* K_α.

Proof. (i) This part was shown in [Eng89, Theorems 2.5.2, 3.2.13 and 6.1.18].

(ii) Each $\rho^{K_\alpha}_{K_\beta}$ is continuous as a restriction of a continuous map. Surjectivity of $\rho^{K_\alpha}_{K_\beta}$ was shown in [Eng89, Corollary 3.2.15]. The relevant equation follows immediately from the definition of an inverse system.

(iii) It was shown in [Eng89, Proposition 2.5.5] that $\mathscr{B}$ is a basis for K_α. Since $|\mathscr{B}_\beta| < 2^\omega$ for each $\beta < 2^\omega$, we have $|\mathscr{B}| < 2^\omega$ whenever $\alpha < 2^\omega$.

(iv) First of all, note that the family

$$\left\{[0,1]^\beta, \pi^\beta_\gamma, \alpha\right\}$$

forms an inverse system and for every $\gamma \leqslant \beta < \alpha$ we have a commutative diagram

$$\begin{array}{ccc} K_\beta & \xrightarrow{\ \phi_\beta\ } & [0,1]^\beta \\ {\scriptstyle \rho^{K_\beta}_{K_\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^\beta_\gamma} \\ K_\gamma & \xrightarrow[\ \phi_\gamma\ ]{} & [0,1]^\gamma \end{array}$$

Define

$$I = \varprojlim_{\beta<\alpha} [0,1]^\beta.$$

By the argument from [Eng89, page 139], the above setup induces a continuous map

$$\underline{\phi_\alpha}\colon K_\alpha \to I,$$

such that the diagram

$$\begin{array}{ccc} K_\alpha & \xrightarrow{\ \underline{\phi_\alpha}\ } & I \\ {\scriptstyle \rho^{K_\alpha}_{K_\beta}}\big\downarrow & & \big\downarrow{\scriptstyle \rho^{I}_{[0,1]^\beta}} \\ K_\beta & \xrightarrow[\ \phi_\beta\ ]{} & [0,1]^\beta \end{array}$$

is commutative for each $\beta < \alpha$. Furthermore, by [Eng89, Theorem 3.2.14], $\underline{\phi_\alpha}$ is surjective.

To complete the construction, note that, by definition of the inverse limit,

$$I = \left\{ (x_\beta)_{\beta<\alpha} \in \prod_{\beta<\alpha} [0,1]^\beta : (x_\beta)_{\beta<\alpha} \text{ is a } \textit{thread}\text{, i.e. } \pi^\beta_\gamma(x_\beta) = x_\gamma \text{ whenever } \gamma \leqslant \beta < \alpha \right\}$$

and so there exists a homeomorphism $\phi\colon I \to [0,1]^\alpha$ with $\rho^I_{[0,1]^\beta} = \pi^\alpha_\beta \phi$. We define

$$\phi_\alpha = \phi \circ \underline{\phi_\alpha}.$$

To show that ϕ_α is irreducible, consider any closed $F \subsetneqq K_\alpha$. Defining $F_\beta = \rho^{K_\alpha}_{K_\beta}(F)$ for each β, note that, by [Eng89, Proposition 2.5.6], the family

$$\left\{F_\beta, \left[\rho^{K_\beta}_{K_\gamma}\right]\Big|_{F_\beta}, \alpha\right\}$$

forms an inverse system with $F = \varprojlim_{\beta<\alpha} F_\beta$.

Let $\beta < \alpha$ be such that $F_\beta \subsetneqq K_\beta$ (we know that such β exists as otherwise we would have $F = \varprojlim_{\beta<\alpha} K_\beta = K_\alpha$). Then, by irreducibility of ϕ_β and commutativity,

$$[0,1]^\beta \supsetneqq \phi_\beta(F_\beta) = \phi_\beta\left(\rho^{K_\alpha}_{K_\beta}(F)\right) = \pi^\alpha_\beta(\phi_\alpha(F)).$$

Since π^α_β is a surjection, this implies that $\phi_\alpha(F) \subsetneqq K_\alpha$.

(v) Let x_0, x_1 be distinct points in K_α. Using the basis fixed in (4.18), for each $i \in \{0,1\}$ there exists $\gamma_i < \alpha$ and $U_i \in \mathscr{B}_{\gamma_i}$ with $x_i \in U_i^{[K_\alpha]}$ and $U_0^{[K_\alpha]} \cap U_1^{[K_\alpha]} = \emptyset$.

Pick any β with $\gamma_0, \gamma_1 \leqslant \beta < \alpha$. Then $\pi^{K_\alpha}_{K_\beta}(x_i) \in U_i^{[K_\beta]}$ and $U_0^{[K_\beta]} \cap U_1^{[K_\beta]} = \emptyset$ which means, in particular, that $\pi^{K_\alpha}_{K_\beta}(x_0) \neq \pi^{K_\alpha}_{K_\beta}(x_1)$. Thus there exists $f \in \mathscr{C}(K_\beta)$ with

$$f\left(\pi^{K_\alpha}_{K_\beta}(x_0)\right) \neq f\left(\pi^{K_\alpha}_{K_\beta}(x_1)\right).$$

By density, we may assume that $f \in D_\beta$ and so $f^{[K_\alpha]}$ is an element of D separating x_0 and x_1.

Finally, note that for any β we have $\chi_{K_\alpha} = \chi_{K_\beta}^{[K_\alpha]} \in D$ and so, by the Stone–Weierstrass theorem [Sem71, Theorem 7.3.8], D is dense in $\mathscr{C}(K_\alpha)$, as required.

(vi) Consider the set

$$\begin{aligned}\Delta = \Delta\left((\vee_{n\in\omega} g_n)^{[K_\alpha]}, (g_n^{[K_\alpha]})_{n\in\omega}\right) &= \left\{x \in K_\alpha : \sum_{n\in\omega} g_n^{[K_\alpha]}(x) \neq (\vee_{n\in\omega} g_n)^{[K_\alpha]}(x)\right\} \\ &= \left\{x \in K_\alpha : \sum_{n\in\omega} g_n(\rho^{K_\alpha}_{K_{\beta_0}}(x)) \neq (\vee_{n\in\omega} g_n)(\rho^{K_\alpha}_{K_{\beta_0}}(x))\right\} \\ &= \left(\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}\left(\Delta(\vee_{n\in\omega} g_n, (g_n)_{n\in\omega})\right). \qquad (4.20)\end{aligned}$$

Suppose that $\overline{\Delta}$ contains a nonempty $B \in \mathscr{B}$ where $\mathscr{B}$ is the basis for K_α defined in (4.18). There exist $\beta < \alpha$ and nonempty $\underline{B} \in \mathscr{B}_\beta$ with

$$B = \underline{B}^{[K_\alpha]}.$$

Two cases are now possible.

Case 1. If $\beta < \beta_0$, then $\underline{B}^{[K_{\beta_0}]}$ is a nonempty open subset of K_{β_0} and, by (4.20),

$$\begin{aligned}\left(\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}\left(\overline{\Delta(\vee_{n\in\omega}g_n,(g_n)_{n\in\omega})}\right) \supseteq \overline{\Delta} &\supseteq B \\ &= \left(\rho^{K_\alpha}_{K_\beta}\right)^{-1}(\underline{B}) \\ &= \left(\rho^{K_{\beta_0}}_{\beta}\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}(\underline{B}) \\ &= \left(\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}(\underline{B}^{[K_{\beta_0}]}).\end{aligned}$$

Since $\rho^{K_\alpha}_{K_{\beta_0}}$ is onto, $\rho^{K_\alpha}_{K_{\beta_0}}\left(\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1} = \mathrm{I}_{K_{\beta_0}}$ and hence the last equation implies that

$$\underline{B}^{[K_{\beta_0}]} \subseteq \overline{\Delta(\vee_{n\in\omega}g_n,(g_n)_{n\in\omega})}.$$

This leads to a contradiction as, by Lemma 4.2.4, $\Delta(\vee_{n\in\omega}g_n,(g_n)_{n\in\omega})$ is nowhere dense.

Case 2. If $\beta_0 \leqslant \beta$, then, by the same type of calculations as in (4.20),

$$\Delta = \left(\rho^{K_\alpha}_{K_\beta}\right)^{-1}\left(\Delta(\vee_{n\in\omega}g_n^{[K_\beta]},(g_n^{[K_\beta]})_{n\in\omega})\right) \tag{4.21}$$

and so

$$\left(\rho^{K_\alpha}_{K_\beta}\right)^{-1}\left(\overline{\Delta(\vee_{n\in\omega}g_n^{[K_\beta]},(g_n^{[K_\beta]})_{n\in\omega})}\right) \supseteq \overline{\Delta} \supseteq B = \left(\rho^{K_\alpha}_{K_\beta}\right)^{-1}(\underline{B}).$$

As before, $\rho^{K_\alpha}_{K_\beta}\left(\rho^{K_\alpha}_{K_\beta}\right)^{-1} = \mathrm{I}_{K_\beta}$ which means that

$$B \subseteq \overline{\Delta(\vee_{n\in\omega}g_n^{[K_\beta]},(g_n^{[K_\beta]})_{n\in\omega})}.$$

However, by Lemma 4.2.4, the set $\Delta(\vee_{n\in\omega}g_n^{[K_\beta]},(g_n^{[K_\beta]})_{n\in\omega})$ is nowhere dense and so we get another contradiction.

Thus, our initial assumption was wrong and Δ is nowhere dense. By Lemma 4.2.4, $(\vee_{n\in\omega}g_n)^{[K_\alpha]}$ is a supremum of $(g_n^{[K_\alpha]})$ in $\mathscr{C}(K_\alpha)$.

(vii) The proof is very similar to the one in (vi). Suppose, for a contradiction, that

$$\bigcup \mathcal{P}^{[K_\alpha]} \subseteq g^{-1}(\{0\}), \quad \bigcup \mathcal{Q}^{[K_\alpha]} \subseteq g^{-1}(\{1\})$$

for some $g \in \mathscr{C}(K_\alpha)$. Then, if D is a dense subset of $\mathscr{C}(K_\alpha)$, there exists $h \in D$ with

$$\bigcup \mathcal{P}^{[K_\alpha]} \subseteq h^{-1}([-1/4,1/4]), \quad \bigcup \mathcal{Q}^{[K_\alpha]} \subseteq h^{-1}([3/4,5/4])$$

Take D to be the set fixed in (4.19). Then there exist $k \in \omega$, $\gamma_i < \alpha$ and $h_i \in D_i$ $(0 \leqslant i \leqslant k)$ such that h is a finite combination of $h_0^{[K_\alpha]},\dots,h_k^{[K_\alpha]}$.

Pick any β with $\gamma_0, \ldots, \gamma_k \leqslant \beta < \alpha$. Then $h_i^{[K_\beta]} \in \mathscr{C}(K_\beta)$ for each i and h is a finite combination of $(h_0^{[K_\beta]})^{[K_\alpha]}, \ldots, (h_k^{[K_\beta]})^{[K_\alpha]}$. Consequently, h has the form

$$h = \underline{h}^{[K_\alpha]}$$

for some $\underline{h} \in \mathscr{C}(K_\beta)$. Two cases are now possible.

Case 1. If $\beta < \beta_0$, then $\underline{h}^{[K_{\beta_0}]} \in \mathscr{C}(K_{\beta_0})$ and

$$\begin{aligned} \left(\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}(\bigcup \mathcal{P}) &= \bigcup \mathcal{P}^{[K_\alpha]} \\ &\subseteq \left(\underline{h}^{[K_\alpha]}\right)^{-1}([-1/4, 1/4]) \\ &= \left(\underline{h} \circ \rho^{K_{\beta_0}}_{K_\beta} \circ \rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}([-1/4, 1/4]) \\ &= \left(\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}\left((\underline{h}^{[K_{\beta_0}]})^{-1}([-1/4, 1/4])\right) \end{aligned}$$

which means that

$$\bigcup \mathcal{P} \subseteq \left(\underline{h}^{[K_{\beta_0}]}\right)^{-1}([-1/4, 1/4]).$$

Similarly,

$$\bigcup \mathcal{Q} \subseteq \left(\underline{h}^{[K_{\beta_0}]}\right)^{-1}([3/4, 5/4]),$$

and so $(\mathcal{P}, \mathcal{Q})$ is split in K_{β_0} which is a contradiction.

Case 2. If $\beta_0 \leqslant \beta$, then

$$\begin{aligned} \left(\rho^{K_\alpha}_{K_\beta}\right)^{-1}\left(\bigcup \mathcal{P}^{[K_\beta]}\right) &= \left(\rho^{K_\alpha}_{K_\beta}\right)^{-1}\left((\rho^{K_\beta}_{K_{\beta_0}})^{-1}(\bigcup \mathcal{P})\right) \\ &= \left(\rho^{K_\alpha}_{K_{\beta_0}}\right)^{-1}\left(\bigcup \mathcal{P}\right) \\ &= \bigcup \mathcal{P}^{[K_\alpha]} \\ &\subseteq h^{-1}([-1/4, 1/4]) \\ &= \left(\rho^{K_\alpha}_{K_\beta}\right)^{-1}\left(\underline{h}^{-1}([-1/4, 1/4])\right) \end{aligned}$$

which means that

$$\bigcup \mathcal{P}^{[K_\beta]} \subseteq \underline{h}^{-1}([-1/4, 1/4]),$$

and similarly,

$$\bigcup \mathcal{Q}^{[K_\beta]} \subseteq \underline{h}^{-1}([3/4, 5/4]).$$

Thus the pair $\left(\mathcal{P}^{[K_\beta]}, \mathcal{Q}^{[K_\beta]}\right)$ is split in K_β which is also a contradiction. □

4.5 Construction of K

Our space K is constructed by transfinite induction.

Notation

Fix a surjection $s\colon \{\alpha : \alpha \text{ is a successor ordinal with } 1 < \alpha < 2^\omega\} \to (2^\omega \backslash \{0\}) \times 2^\omega$ such that if $s(\alpha) = (\eta, \zeta)$, then $\eta < \alpha$.

Inductive construction

We are going to construct

(i) a transfinite sequence of topological spaces $(K_\alpha)_{1 \leqslant \alpha < 2^\omega}$ such that for each α

- (i.a) K_α is compact, Hausdorff and connected,
- (i.b) weight$(K_\alpha) < 2^\omega$ and we can fix a basis $\mathscr{B}_\alpha$ of K_α such that $|\mathscr{B}_\alpha| < 2^\omega$ and $\mathscr{B}_\beta^{[K_\alpha]} \subseteq \mathscr{B}_\alpha$ whenever $\beta < \alpha$,
- (i.c) density$(\mathscr{C}(K_\alpha)) < 2^\omega$,
- (i.d) there exists a continuous irreducible surjection $\phi_\alpha\colon K_\alpha \twoheadrightarrow [0,1]^\alpha$,
- (i.e) for any $\beta < \alpha$ there exists a continuous surjection $\rho^{K_\alpha}_{K_\beta}\colon K_\alpha \twoheadrightarrow K_\beta$ such that

$$\rho^{K_\beta}_{K_\gamma}\rho^{K_\alpha}_{K_\beta} = \rho^{K_\alpha}_{K_\gamma} \quad \text{and} \quad \rho^{K_\beta}_{K_\beta} = \mathrm{I}_{K_\beta}$$

  whenever $\gamma \leqslant \beta < \alpha$,
- (i.f) with notation as above, the diagram

$$\begin{array}{ccc} K_\alpha & \xrightarrow{\ \phi_\alpha\ } & [0,1]^\alpha \\ {\scriptstyle \rho^{K_\alpha}_{K_\beta}}\downarrow & & \downarrow{\scriptstyle \pi^\alpha_\beta} \\ K_\beta & \xrightarrow[\ \phi_\beta\ ]{} & [0,1]^\beta \end{array}$$

  is commutative for any $\beta < \alpha$,
- (i.g) if $\beta < \alpha$ and (g_n) is a disjoint sequence in $\mathscr{C}(K_\beta)$ such that range$(g_n) \subseteq [0,1]$ for each n and (g_n) has a supremum in $\mathscr{C}(K_\beta)$, then the supremum of (g_n) is preserved in $\mathscr{C}(K_\alpha)$,

(i.h) we fix an enumeration of all the quadruplets

$((f_n(\alpha,\zeta))_{n\in\omega}, (U_n(\alpha,\zeta))_{n\in\omega}, (V_n(\alpha,\zeta))_{n\in\omega}, (W_n(\alpha,\zeta))_{n\in\omega})$ $(\zeta < 2^\omega)$ with the property that

- $(f_n(\alpha,\zeta))_{n\in\omega}$ is a disjoint sequence in $\mathscr{C}(K_\alpha)$ and $\mathrm{range}(f_n) \subseteq [0,1]$ for all n,
- each of the remaining sequences is disjoint and consists of elements of $\mathscr{B}_\alpha$,
- all $U_n(\alpha,\zeta)$ are nonempty,
- $\mathrm{supp}(f_m(\alpha,\zeta)) \cap U_n(\alpha,\zeta) = \emptyset$ for all $m,n\in\omega$ and $\zeta < 2^\omega$,

(i.i) using the enumeration from (i.h), if $\alpha = \beta+1$ with $\beta > 1$, then

- $K_\alpha = K_\beta(\tau)\times[0,1]$, where $K_\beta(\tau)$ is the extension of K_β by $(f_n(s(\alpha))^{[K_\beta]})_{n\in\tau}$ and is strong,

(i.j) if α is a limit ordinal, then

- $K_\alpha = \varprojlim_{\beta<\alpha} K_\beta$,

(ii) arrays $(\mathcal{P}_{\alpha,I}, \mathcal{Q}_{\alpha,I})$ with $\alpha < 2^\omega$ and $I \in \{0,1,[0,1/3],[2/3,1]\}$ with the property that

(ii.a) if $\alpha = \beta+1$ with $\beta \geqslant 1$ then

- $(\mathcal{P}_{\alpha,0}, \mathcal{Q}_{\alpha,0}) = \big((U_n(s(\alpha))^{[K_\alpha]})_{n\in\tau}, (U_n(s(\alpha))^{[K_\alpha]})_{n\notin\tau}\big)$,
- $(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1}) = \big((V_n(s(\alpha))^{[K_\alpha]})_{n\in\omega}, (W_n(s(\alpha))^{[K_\alpha]})_{n\in\omega}\big)$ if this pair cannot be split in K_α, or $(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1}) = (\emptyset,\emptyset)$ otherwise,
- $\big(\mathcal{P}_{\alpha,[0,1/3]}, \mathcal{Q}_{\alpha,[0,1/3]}\big) = \Big(\mathcal{P}_{\beta,1}^{[K_\beta(\tau)]} \times [0,1/3], \mathcal{Q}_{\beta,1}^{[K_\beta(\tau)]} \times [0,1/3]\Big)$,
- $\big(\mathcal{P}_{\alpha,[2/3,1]}, \mathcal{Q}_{\alpha,[2/3,1]}\big) = \Big(\mathcal{P}_{\beta,1}^{[K_\beta(\tau)]} \times [2/3,1], \mathcal{Q}_{\beta,1}^{[K_\beta(\tau)]} \times [2/3,1]\Big)$,

(ii.b) if α is a limit ordinal, then

- $(\mathcal{P}_{\alpha,I}, \mathcal{Q}_{\alpha,I}) = (\emptyset,\emptyset)$ for each I,

(ii.c) $(\mathcal{P}_{\alpha,I}, \mathcal{Q}_{\alpha,I})$ forms a forbidden splitting in K_α unless it is equal to $(\emptyset,\emptyset)$,

(ii.d) If $\beta < \alpha$ and $(\mathcal{P}_{\beta,I}, \mathcal{Q}_{\beta,I})$ is forbidden in K_β, it remains forbidden in K_α.

Note 4.5.1. Existence of the enumeration in (i.h) follows from (i.b)–(i.c) and does not need to be taken care of separately.

Base case

Define

$$K_1 = [0,1],$$

and for each $I \in \{0, 1, [0, 1/3], [2/3, 1]\}$ set

$$(\mathcal{P}_{1,I}, \mathcal{Q}_{1,I}) = (\emptyset, \emptyset) .$$

Inductive step

Suppose that for some $\alpha < 2^\omega$ we have constructed $(K_\beta)_{\beta<\alpha}$ and $((\mathcal{P}_{\beta,I}, \mathcal{Q}_{\beta,I}))$ satisfying conditions (i.a)–(ii.d).

Case 1. α is a successor ordinal

Suppose that $\alpha = \beta + 1$ for some ordinal β with $1 \leqslant \beta < 2^\omega$.

To construct K_α, we start with taking care of property (K). Suppose that $s(\alpha) = (\eta, \zeta)$. Then $1 \leqslant \eta \leqslant \beta$ which means that η^{th} stage of our construction has already been completed. Using the enumeration fixed in (i.h), for each $n \in \omega$ we define

$$f_n = f_n(\eta, \zeta)^{[K_\beta]}, \quad U_n = U_n(\eta, \zeta)^{[K_\beta]},$$

$$V_n = V_n(\eta, \zeta)^{[K_\beta]}, \quad W_n = W_n(\eta, \zeta)^{[K_\beta]}.$$

Then (f_n) is a disjoint sequence in $\mathscr{C}(K_\beta)$ with $\text{range}(f_n) \subseteq [0,1]$ for each n while, by (i.b), each of (U_n), (V_n), (W_n) is a disjoint sequence in $\mathscr{B}_\beta$. In addition, we also have $U_n \neq \emptyset$ and $\text{supp}(f_m) \cap U_n = \emptyset$ for all m, n.

At this stage there are at most $4\beta < 2^\omega$ splittings which are forbidden in K_β and need to be preserved in K_α. Hence, by Proposition 4.4.3, there exists an infinite $\tau \subseteq \omega$ such that

- the space $K_\beta(\tau)$ satisfies the conditions (i.a)–(i.f),
- if (g_n) is a disjoint sequence in $\mathscr{C}(K_\beta)$ such that $\text{range}(g_n) \subseteq [0,1]$ for each n and (g_n) has a supremum in $\mathscr{C}(K_\beta)$, then this supremum is preserved in $\mathscr{C}(K_\beta(\tau))$,
- a supremum of $(f_n^{[K_\beta(\tau)]})$ exists in $\mathscr{C}(K_\beta(\tau))$ and is equal to $f_\tau^{[K_\beta(\tau)]}$,
- if, for some $\gamma \leqslant \beta$ and $I \in \{0, 1, [0, 1/3], [2/3, 1]\}$, the splitting $\left(\mathcal{P}^{[\beta]}_{\gamma,I}, \mathcal{Q}^{[\beta]}_{\gamma,I}\right)$ is forbidden in K_β, it remains forbidden in $K_\beta(\tau)$,

- the pair $\left((U_n^{[K_\beta(\tau)]})_{n\in\tau}, (U_n^{[K_\beta(\tau)]})_{n\notin\tau}\right)$ cannot be split in $K_\beta(\tau)$.

With τ as above we define

$$K_\alpha = K_\beta(\tau) \times [0,1]. \tag{4.22}$$

Appealing to Proposition 4.4.4,

- K_α satisfies the conditions (i.a)–(i.f) and (i.h)–(i.i),
- if (g_n) is a disjoint sequence in $\mathscr{C}(K_\beta(\tau))$ such that $\mathrm{range}(g_n) \subseteq [0,1]$ for each n and (g_n) has a supremum in $\mathscr{C}(K_\beta(\tau))$, then the supremum is preserved in K_α,
- if $(\mathcal{P}, \mathcal{Q})$ is a forbidden splitting in $K_\beta(\tau)$, it remains forbidden in K_α,
- if $(\mathcal{P}, \mathcal{Q})$ is a forbidden splitting in $K_\beta(\tau)$, then both $(\mathcal{P} \times [0,1/3], \mathcal{Q} \times [0,1/3])$ and $(\mathcal{P} \times [2/3,1], \mathcal{Q} \times [2/3,1])$ are forbidden in K_α.

To finish the construction, we define

$$(\mathcal{P}_{\alpha,0}, \mathcal{Q}_{\alpha,0}) = \left((U_n^{[K_\alpha]})_{n\in\tau}, (U_n^{[K_\alpha]})_{n\notin\tau}\right),$$

$$\left(\mathcal{P}_{\alpha,[0,1/3]}, \mathcal{Q}_{\alpha,[0,1/3]}\right) = \left(\mathcal{P}_{\beta,1}^{[K_\beta(\tau)]} \times [0,1/3], \mathcal{Q}_{\beta,1}^{[K_\beta(\tau)]} \times [0,1/3]\right),$$

$$\left(\mathcal{P}_{\alpha,[2/3,1]}, \mathcal{Q}_{\alpha,[2/3,1]}\right) = \left(\mathcal{P}_{\beta,1}^{[K_\beta(\tau)]} \times [2/3,1], \mathcal{Q}_{\beta,1}^{[K_\beta(\tau)]} \times [2/3,1]\right)$$

and

$$(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1}) = \begin{cases} \left((V_n^{[K_\alpha]})_{n\in\omega}, (W_n^{[K_\alpha]})_{n\in\omega}\right) & \text{if } \left((V_n^{[K_\alpha]})_{n\in\omega}, (W_n^{[K_\alpha]})_{n\in\omega}\right) \text{ cannot be split in } K_\alpha \\ (\emptyset, \emptyset) & \text{otherwise.} \end{cases}$$

By construction, the splittings $(\mathcal{P}_{\alpha,i}, \mathcal{Q}_{\alpha,i})$ satisfy the conditions (ii.a)–(ii.c). Furthermore, if for some $\gamma < \alpha$ and $I \in \{0, 1, [0,1/3], [2/3,1]\}$ the splitting $(\mathcal{P}_{\gamma,i}, \mathcal{Q}_{\gamma,i})$ is forbidden in K_γ, by the inductive assumption, it remains forbidden in K_β. Thus it remains forbidden in $K_\beta(\tau)$ and hence also in K_α. Consequently, K_α satisfies the condition (ii.d).

Finally, suppose that for some $\gamma \leqslant \alpha$ we have a disjoint sequence $(g_n) \subseteq \mathscr{C}(K_\gamma)$ such that $\mathrm{range}(g_n) \subseteq [0,1]$ for each n and (g_n) has a supremum in $\mathscr{C}(K_\gamma)$. By the inductive assumption, the supremum is preserved in K_β. Thus it is preserved in $K_\beta(\tau)$ and hence in K_α. Consequently, K_α satisfies the condition (i.g).

Case 2. α is a limit ordinal

In this case we define

$$K_\alpha = \varprojlim_{\beta<\alpha} K_\beta$$

and for each $I \in \{0, 1, [0, 1/3], [2/3, 1]\}$ we set

$$(\mathcal{P}_{\alpha,I}, \mathcal{Q}_{\alpha,I}) = (\emptyset, \emptyset)\,.$$

Proposition 4.4.5 guarantees that the conditions (i.a)–(ii.d) are satisfied at this stage.

Construction of K

Once the construction of $(K_\alpha)_{\alpha<2^\omega}$ has been completed, we define

$$K = \varprojlim_{\alpha<2^\omega} K_\alpha.$$

4.6 Properties of K

Proposition 4.6.1 (P1)**.** *K is separable.*

Proof. By Proposition 4.4.5 (iv), there exists an irreducible surjection $\phi_{2^\omega}\colon K \to [0,1]^{2^\omega}$. However, $[0,1]^{2^\omega}$ is separable (see e.g. [Dug66, p. 175, Theorem 7.2]) and so the result follows from Theorem 3.2.14. □

Proposition 4.6.2 (P2)**.** *K is weakly Koszmider.*

We start with showing an intermediate result.

Lemma 4.6.3. *Let K be as above and consider any $f \in \mathscr{C}(K)$. Then there exist $\beta < 2^\omega$ and $\underline{f} \in \mathscr{C}(K_\beta)$ with*

$$f = \underline{f}^{[K]}.$$

Proof. Since K is a closed subset of $\prod_{\beta<\alpha} K_\beta$, by the Tietze extension theorem [Wil70, Theorem 15.8], there exists a function $F \in \mathscr{C}(\prod_{\beta<2^\omega} K_\beta)$ with

$$[F]|_K = f.$$

By Mibu's Theorem [Mib44], F depends on countably many coordinates, hence, by König's Lemma [Kun80, p. 34, Lemma 10.40], there exist $\beta < 2^\omega$ and $\underline{F} \in \mathscr{C}(\prod_{\gamma\leqslant\beta} K_\gamma)$ such that

$$F = \underline{F} \circ \rho^{\prod_{\gamma<2^\omega} K_\gamma}_{\prod_{\gamma\leqslant\beta} K_\gamma},$$

and so, since K consists of threads,

$$f = (\underline{F} \circ \rho)^{[K]},$$

where the function ρ is defined in the following way

$$\rho\colon K_\beta \to \prod_{\gamma\leqslant\beta} K_\gamma, \quad x \mapsto \left(\rho^{K_\beta}_{K_\gamma}(x)\right)_{\gamma\leqslant\beta}.$$

□

Proof of Proposition 4.6.2. By Theorem 4.3.2, it is sufficient to show that K has the property (K).

Suppose that (f_n) is a disjoint sequence in $\mathscr{C}(K)$ with range$(f_n) \subseteq [0,1]$ for each n and (U_n) is a disjoint sequence of nonempty open subsets of K with $\operatorname{supp}(f_m) \cap U_n = \emptyset$ for all m, n. By Proposition 4.4.5 and Lemma 4.6.3, for each n there exist $\alpha_n, \beta_n < 2^\omega$, $\underline{f}_n \in \mathscr{C}(K_{\alpha_n})$ and nonempty $\underline{U}_n \in \mathscr{B}_{\beta_n}$ (where $\mathscr{B}_{\beta_n}$ is the basis fixed in (i.b)) such that

$$\underline{f}_n^{[K]} = f_n, \quad \underline{U}_n^{[K]} \subseteq U_n.$$

By König's Lemma [Kun80, p. 34, Lemma 10.40], there exists η with $1 \leqslant \eta < 2^\omega$ and $\alpha_n, \beta_n < \eta$ for all n.

Consider now the sequence $(\underline{f}_n^{[K_\eta]})$. Take any $x \in K_\eta$. By surjectivity of $\rho^K_{K_\eta}$, there exists $y \in K$ with $x = \rho^K_{K_\eta}(y)$. Then for any $m \neq n$ we have

$$\begin{aligned}
\underline{f}_m^{[K_\eta]}(x) \cdot \underline{f}_n^{[K_\eta]}(x) &= \Big((\underline{f}_m \circ \rho^{K_\eta}_{K_{\alpha_m}})(\rho^K_{K_\eta}(y))\Big) \cdot \Big((\underline{f}_n \circ \rho^{K_\eta}_{K_{\alpha_n}})(\rho^K_{K_\eta}(y))\Big) \\
&= \Big((\underline{f}_m \circ \rho^{K}_{K_{\alpha_m}})(y)\Big) \cdot \Big((\underline{f}_n \circ \rho^{K}_{K_{\alpha_n}})(y)\Big) \\
&= f_m(y) \cdot f_n(y) \\
&= 0,
\end{aligned}$$

and so $(\underline{f}_n^{[K_\eta]})$ is a disjoint sequence in $\mathscr{C}(K_\eta)$ with range$(f_n) \subseteq [0,1]$ for each n. Similarly, $(\underline{U}_n^{[K_\eta]})$ is a disjoint sequence in K_η and $\operatorname{supp}(\underline{f}_m^{[K_\eta]}) \cap \underline{U}_n^{[K_\eta]} = \emptyset$ for all m, n. Finally, by (i.b), $(\underline{U}_n^{[K_\eta]}) \subseteq \mathscr{B}_\eta$. Using the enumeration fixed in (i.h), there exists $\zeta < 2^\omega$ such that

$$\underline{f}_n^{[K_\eta]} = f_n(\eta,\zeta), \quad \underline{U}_n^{[K_\eta]} = U_n(\eta,\zeta)$$

for all n. Take now any successor ordinal $\alpha < 2^\omega$ with $s(\alpha) = (\eta, \zeta)$. By construction, there exists $\tau \subseteq \omega$ such that

- a supremum of $\left((\underline{f}_n^{[K_\eta]})^{[K_\alpha]}\right)_{n\in\tau}$ exists in $\mathscr{C}(K_\alpha)$,
- the pair $\left(\left((\underline{U}_n^{[K_\eta]})^{[K_\alpha]}\right)_{n\in\tau}, \left((\underline{U}_n^{[K_\eta]})^{[K_\alpha]}\right)_{n\notin\tau}\right)$ forms the splitting $(\mathcal{P}_{\alpha,0}, \mathcal{Q}_{\alpha,0})$ and is forbidden in K_α.

By construction, the supremum of $\left((\underline{f}_n^{[K_\eta]})^{[K_\alpha]}\right)_{n\in\tau}$ is preserved in $\mathscr{C}(K_{\alpha'})$ whenever $\alpha \leqslant \alpha' < 2^\omega$, which, by Proposition 4.4.5, means that

$$\mathscr{C}(K) \ni \vee_{n\in\tau}\left((\underline{f}_n^{[K_\eta]})^{[K_\alpha]}\right)^{[K]} = \vee_{n\in\tau}\underline{f}_n^{[K]} = \vee_{n\in\tau}f_n$$

giving the first condition of property (K).

Similarly, the forbidden splitting $(\mathcal{P}_{\alpha,0}, \mathcal{Q}_{\alpha,0})$ is preserved in $K_{\alpha'}$ whenever $\alpha \leqslant \alpha' < 2^\omega$ which, by Proposition 4.4.5, means that the pair

$$\begin{aligned}\left(\mathcal{P}_{\alpha,0}^{[K]}, \mathcal{Q}_{\alpha,0}^{[K]}\right) &= \left(\left(\left((\underline{U}_n^{[K_\eta]})^{[K_\alpha]}\right)^{[K]}\right)_{n\in\tau}, \left(\left((\underline{U}_n^{[K_\eta]})^{[K_\alpha]}\right)^{[K]}\right)_{n\notin\tau}\right)\\ &= \left((\underline{U}_n^{[K]})_{n\in\tau}, (\underline{U}_n^{[K]})_{n\notin\tau}\right),\end{aligned}$$

and consequently, the pair $((U_n)_{n\in\tau}, (U_n)_{n\notin\tau})$, cannot be split in K.

The above two conclusions show that K has property (K). □

Proposition 4.6.4 (P3)**.** *K has no open butterflies.*

Proof. Consider the following basis of K

$$\mathscr{B} = \bigcup_{\alpha<2^\omega}\{B^{[K]} : B \in \mathscr{B}_\alpha\},$$

where $\mathscr{B}_\alpha$ are the bases fixed in (i.b).

Let V, W be open subsets of K with

$$\overline{V} \cap \overline{W} \neq \emptyset.$$

By Lemma 3.6.4, there exist disjoint $(V_n), (W_n) \subseteq \mathscr{B}$ with $\overline{\bigcup V_n} = \overline{V}$ and $\overline{\bigcup W_n} = \overline{W}$. Thus

$$\overline{\bigcup V_n} \cap \overline{\bigcup W_n} \neq \emptyset.$$

We now proceed exactly as in the previous part. For each n there exist $\alpha_n, \beta_n < 2^\omega$, $\underline{V}_n \in \mathscr{B}_{\alpha_n}$ and $\underline{W}_n \in \mathscr{B}_{\beta_n}$ such that

$$V_n = \underline{V}_n^{[K]}, \quad W_n = \underline{W}_n^{[K]}.$$

By König's Lemma [Kun80, p. 34, Lemma 10.40], we can find η with $1 \leqslant \eta < 2^\omega$ and $\alpha_n, \beta_n < \eta$ for all n. Then $(\underline{V}_n^{[K_\eta]}), (\underline{W}_n^{[K_\eta]}) \subseteq \mathscr{B}_\eta$ and are disjoint. Consequently, using the enumeration fixed in (i.h), there exists $\zeta < 2^\omega$ such that for all n we have

$$\underline{V}_n^{[K_\eta]} = V_n(\eta, \zeta), \quad \underline{W}_n^{[K_\eta]} = W_n(\eta, \zeta).$$

Let now $\alpha < 2^\omega$ be a successor ordinal with $s(\alpha) = (\eta, \zeta)$. Then $\alpha_n, \beta_n < \eta < \alpha$ for each n and so

$$\begin{aligned}
\overline{\bigcup \underline{V}_n^{[K_\alpha]}} \cap \overline{\bigcup \underline{W}_n^{[K_\alpha]}} &= \rho^K_{K_\alpha}\left(\left(\rho^K_{K_\alpha}\right)^{-1}\left(\overline{\bigcup \underline{V}_n^{[K_\alpha]}} \cap \overline{\bigcup \underline{W}_n^{[K_\alpha]}}\right)\right) \\
&\supseteq \rho^K_{K_\alpha}\left(\overline{\bigcup \left(\rho^K_{K_\alpha}\right)^{-1}\left(\underline{V}_n^{[K_\alpha]}\right)} \cap \overline{\bigcup \left(\rho^K_{K_\alpha}\right)^{-1}\left(\underline{W}_n^{[K_\alpha]}\right)}\right) \\
&= \rho^K_{K_\alpha}\left(\overline{\bigcup \underline{V}_n^{[K]}} \cap \overline{\bigcup \underline{W}_n^{[K]}}\right) \\
&= \rho^K_{K_\alpha}\left(\overline{\bigcup V_n} \cap \overline{\bigcup W_n}\right) \\
&\neq \emptyset,
\end{aligned}$$

and so $\left(\underline{V}_n^{[K_\alpha]}, \underline{W}_n^{[K_\alpha]}\right)$ forms the forbidden splitting $(\mathcal{P}_{\alpha,1}, \mathcal{Q}_{\alpha,1})$ in K_α. Following the notation from (4.22), for each $I \in \{[0,1/3],[2/3,1]\}$ the pair $\left((\underline{V}_n^{[K_\alpha(\tau)]} \times I), (\underline{W}_n^{[K_\alpha(\tau)]} \times I)\right)$ forms the forbidden splitting $(\mathcal{P}_{\alpha+1,I}, \mathcal{Q}_{\alpha+1,I})$ in $K_{\alpha+1}$ which is then preserved in $K_{\alpha'}$ whenever $\alpha + 1 \leqslant \alpha' < 2^\omega$. Proposition 4.4.5 now ensures that it remains forbidden in K. In particular, there exists

$$\begin{aligned}
x_I &\in \overline{\bigcup (\underline{V}_n^{[K_\alpha(\tau)]} \times I)^{[K]}} \cap \overline{\bigcup (\underline{W}_n^{[K_\alpha(\tau)]} \times I)^{[K]}} \\
&\subseteq \overline{\bigcup (\underline{V}_n^{[K_\alpha(\tau)]} \times [0,1])^{[K]}} \cap \overline{\bigcup (\underline{W}_n^{[K_\alpha(\tau)]} \times [0,1])^{[K]}} \\
&= \overline{\bigcup (\underline{V}_n^{[K_{\alpha+1}]})^{[K]}} \cap \overline{\bigcup (\underline{W}_n^{[K_{\alpha+1}]})^{[K]}} \\
&= \overline{\bigcup \underline{V}_n^{[K]}} \cap \overline{\bigcup \underline{W}_n^{[K]}} \\
&= \overline{\bigcup V_n} \cap \overline{\bigcup W_n} \\
&= \overline{V} \cap \overline{W}.
\end{aligned}$$

Finally, note that

$$\begin{aligned}
& \overline{\bigcup(\underline{V}_n^{[K_\alpha(\tau)]} \times [0,1/3])^{[K]}} \cap \overline{\bigcup(\underline{V}_n^{[K_\alpha(\tau)]} \times [2/3,1])^{[K]}} \\
= \;& \overline{\left(\rho^K_{K_{\alpha+1}}\right)^{-1}\left(\bigcup(\underline{V}_n^{[K_\alpha(\tau)]} \times [0,1/3])\right)} \cap \overline{\left(\rho^K_{K_{\alpha+1}}\right)^{-1}\left(\bigcup(\underline{V}_n^{[K_\alpha(\tau)]} \times [2/3,1])\right)} \\
\subseteq \;& \left(\rho^K_{K_{\alpha+1}}\right)^{-1}\left(\overline{\bigcup \underline{V}_n^{[K_\alpha(\tau)]}} \times [0,1/3] \cap \overline{\bigcup \underline{V}_n^{[K_\alpha(\tau)]}} \times [2/3,1]\right) \\
= \;& \emptyset,
\end{aligned}$$

which means that

$$x_{[0,1/3]} \neq x_{[2/3,1]},$$

and so $\overline{V} \cap \overline{W}$ contains at least two points. □

Proposition 4.6.5 (P4)**.** *K is connected.*

Proof. This was shown in part (i) of Proposition 4.4.5. □

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